Tools of Dynamical Systems Enable Computers to Interpret Crowd Behavior

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A Gauss 2012 Tutorial

Traffic Flow, Crowd Flow, Fluid Flow

Contents

- Crowd Behavior Recognition
- Segmentation of Motion
- Chaotic Dynamical Systems

Mathematical Tools

- Lagrangian approach to fluid dynamics
- Takens' Theorem: reconstruction of a chaotic system

BLOOD SWEAT TEARS

Recall

Consider a 2-D dynamical system

$$\dot{w} = F(w) \quad \Longleftrightarrow \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}.$$

We can understand the flow by linearization $\dot{z} = J_F(w^*)z$.

- w^* if a fixed point, i.e. $F(w^*) = 0$
- $J_F(w^*)$ is the Jacobian matrix

$$J_F(w^*) = \begin{pmatrix} u_x(w^*) & u_y(w^*) \\ v_x(w^*) & v_y(w^*) \end{pmatrix}$$

Recall

The solutions of a linear system of differential equations $\dot{z} = Az$ are determined by the eigenvalues of A, given by λ_1, and λ_2.

If A is diagonalizable, then $A = P\Lambda P^{-1}$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Thus the system of equations becomes

$$\dot{z} = P\Lambda P^{-1} z \quad \Longleftrightarrow \quad \begin{aligned} \dot{\phi} &= \Lambda\phi \\ \phi &= P^{-1}z \end{aligned}$$

Behavior Recognition

Linear Stability Analysis for Dynamical Systems

- Flow is described by $\dot{w} = F(w)$, with $w = (x, y)^T$ and F is obtained from the optical flow
- Linearizing about fixed points w^* gives $\dot{z} = J_F(w^*)z$.
- Behavior near the fixed points is characterized by

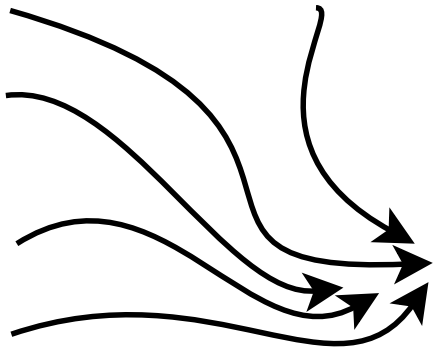
$$\lambda^2 - \tau\lambda + \Delta = 0 \quad \Rightarrow \quad \lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

$\tau = \lambda_1 + \lambda_2$ is the trace and $\Delta = \lambda_1\lambda_2$ is the determinant

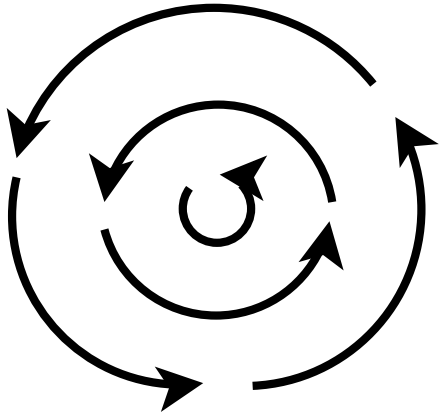
B. Solmaz, B.E. Moore, and M. Shah, Identifying Behaviors in Crowded Scenes through Stability Analysis for Dynamical Systems, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 2012.

Behavior Recognition

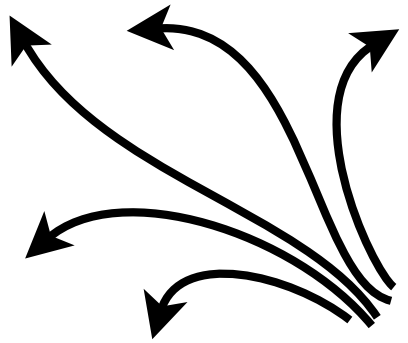
Stability/Behaviors



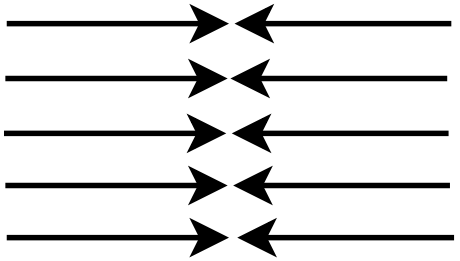
$\Delta > 0, \tau < 0$



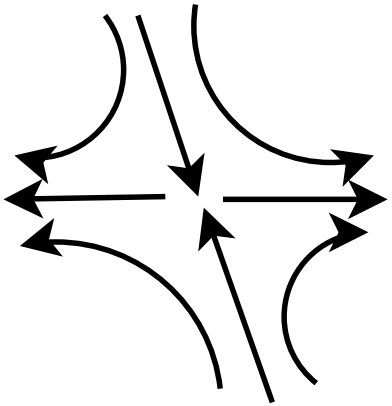
$\Delta > 0, \tau = 0$



$\Delta > 0, \tau > 0$



$\Delta = 0$

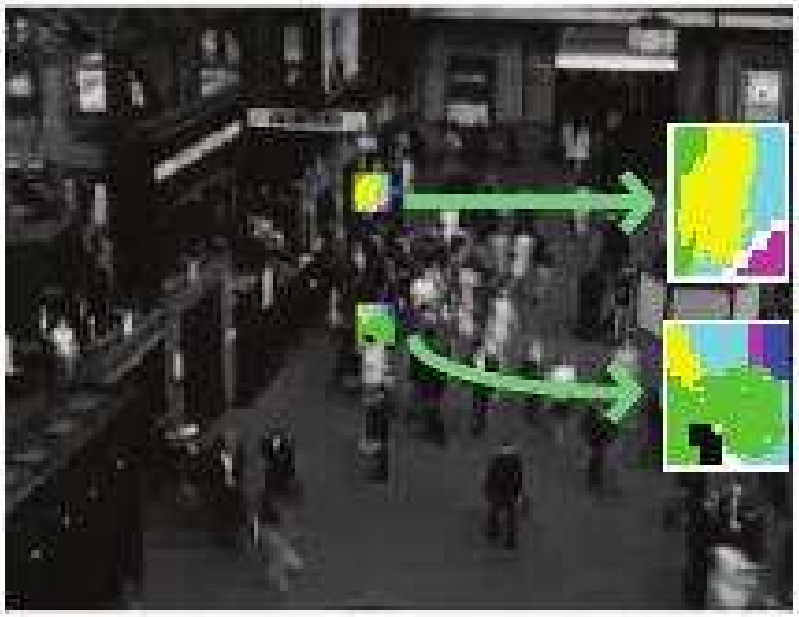


$\Delta < 0$

Behavior Recognition Method

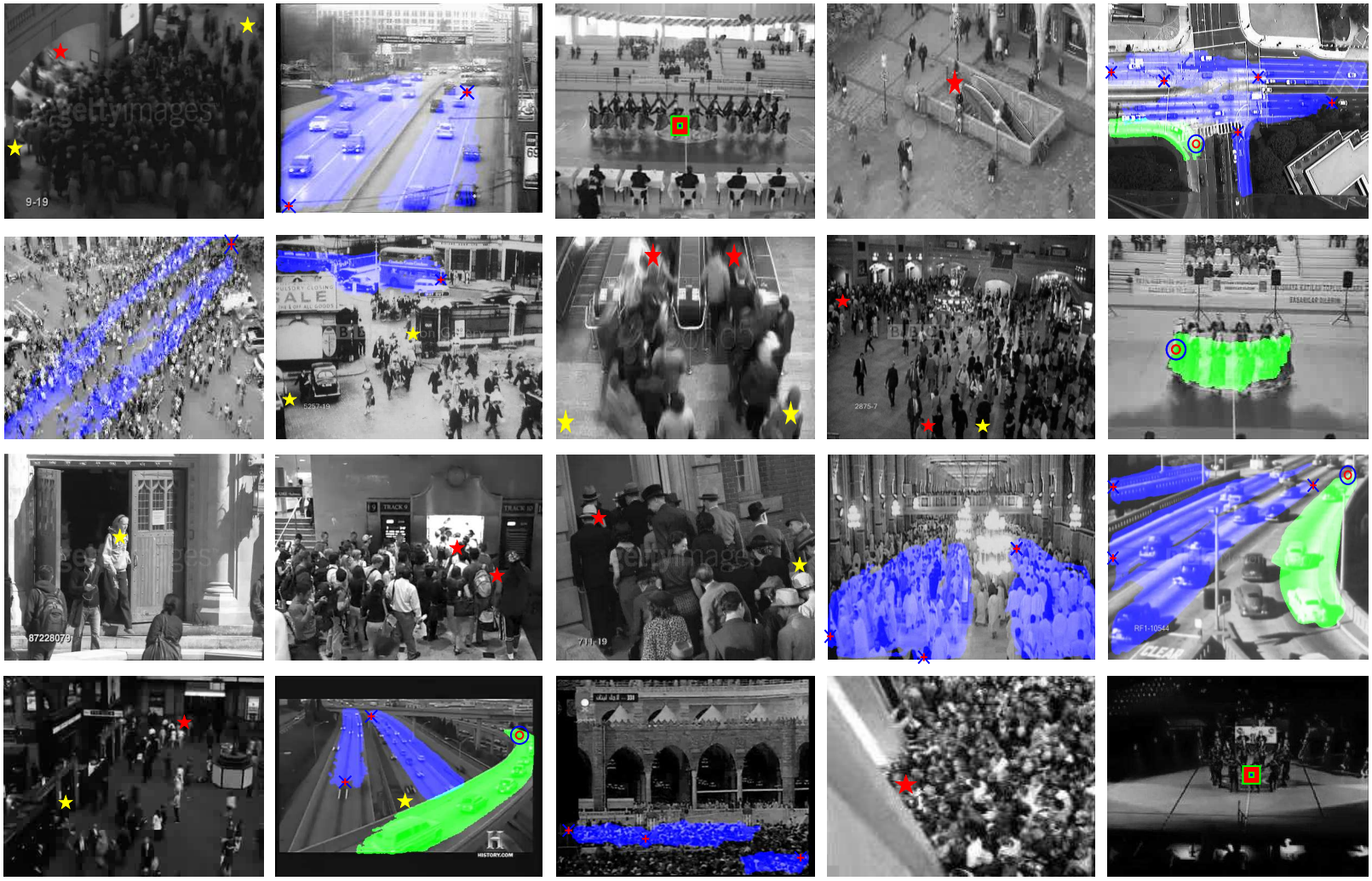
Behavior Recognition Method

Behavior Recognition Method



Behavior Recognition

Results



★ : Bottleneck ★ : Fountainhead ◎ : Arch/Ring ✕ : Lane ■ : Blocking

Behavior Recognition Results

Table 1: Crowd Behavior Detection Results

Behavior	Total # of Behaviors	# of Detections	# of Missed	# of False
Lane	61	52	9	11
Blocking	3	3	0	0
Bottleneck	20	16	4	3
Fountainhead	29	23	7	5
Arch/Ring	28	23	5	6

Lyapunov Exponents for Flow Segmentation

- Track movement of pixels as particle trajectories.
- Compute the distance between neighboring particles at the end of a trajectory.
- Particles that stay close together are part of the same coherent flow pattern.
- Particles that diverge belong to separate coherent flow patterns.
- Tool: Finite Time Lyapunov Exponent (FTLE)

S. Ali and M. Shah, A Lagrangian Particle Dynamics Approach for Crowd Flow Segmentation and Stability Analysis, IEEE CVPR, 2007.

B.E. Moore, S. Ali, R. Mehran, and M. Shah, Visual Crowd Surveillance through a Hydrodynamics Lens, *Communications of the ACM*, 54(12):64-73, December 2011.

Example Video Sequence

Particle Advection

- Every pixel has a position (x, y).
- Opical flow provides the velocities (u, v) at each pixel.

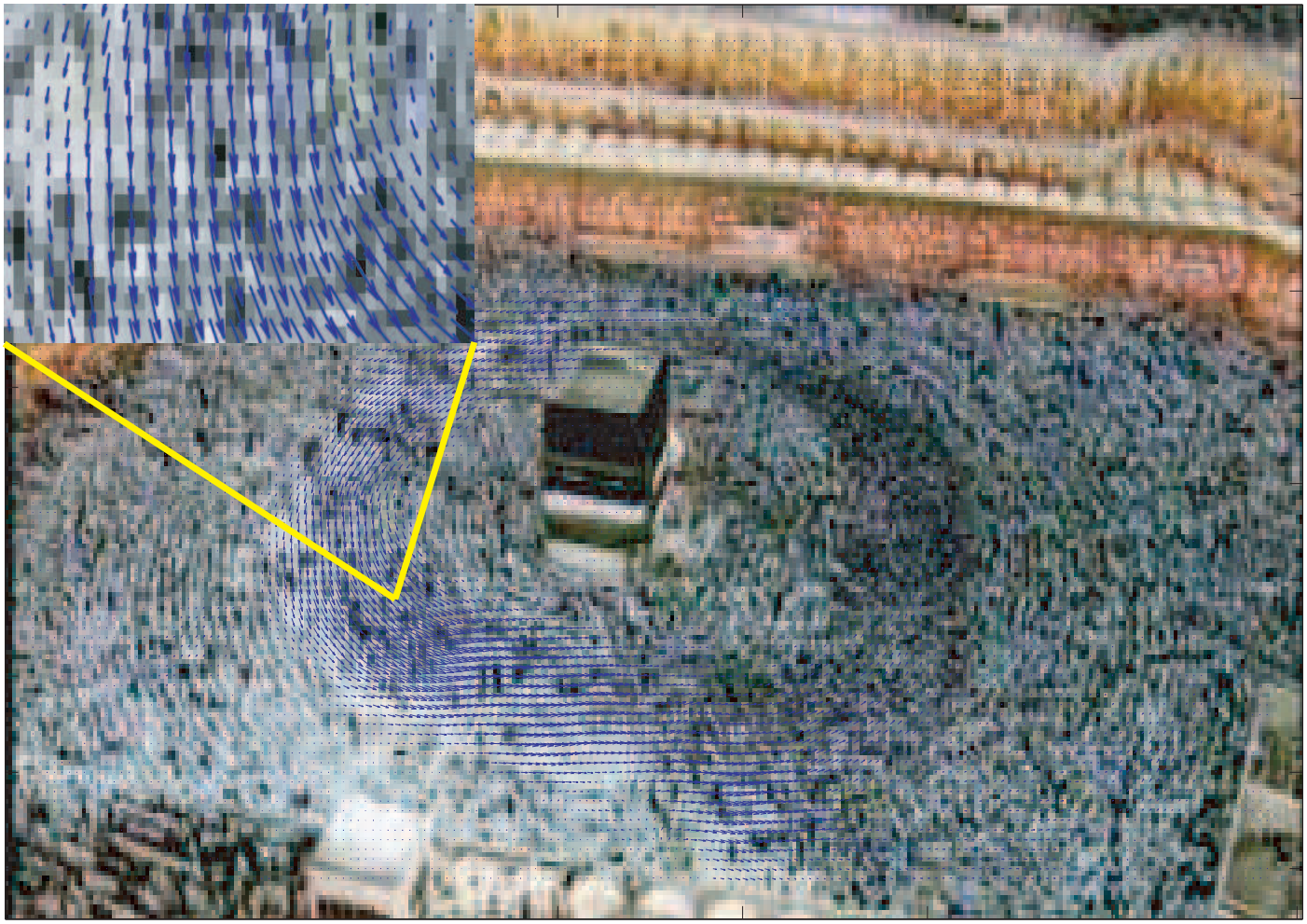
$$\frac{dx}{dt} = u(x, y, t), \qquad \frac{dy}{dt} = v(x, y, t)$$

- Initial Condition: Overlay scene with a grid of particles
- Particles transported to new coordinates by time-stepping.

$$x(t+1) = x(t) + u(x(t), y(t), t), \quad y(t+1) = x(t) + v(x(t), y(t), t)$$

Performing computations over a time interval ($\approx$ 60 frames) gives particle trajectories describing the motion in the scene.

Computed Particle Trajectories for Video Sequence



Computing the FTLE

The jth pair of nearest neighbors diverge at a rate L.

$$d_j(t_i) \approx k_j e^{Lt_i} \quad \Longleftrightarrow \quad \ln d_j(t_i) \approx \ln k_j + Lt_i$$

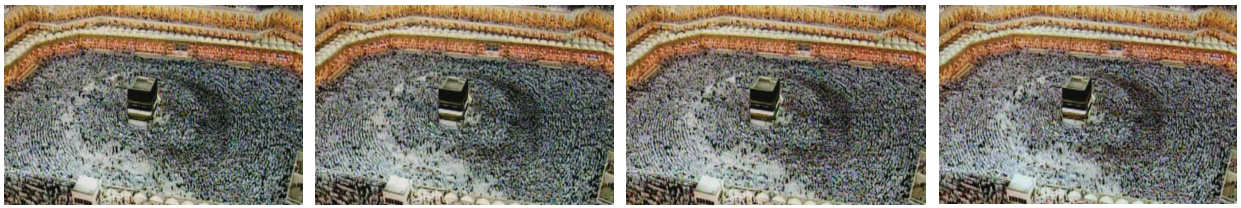
- $t_i = i\Delta t$ and Δt is the sampling period
- k_j is the initial separation
- $d_j(t_i)$ denotes the distance between the jth pair of nearest neighbors after i discrete time steps.

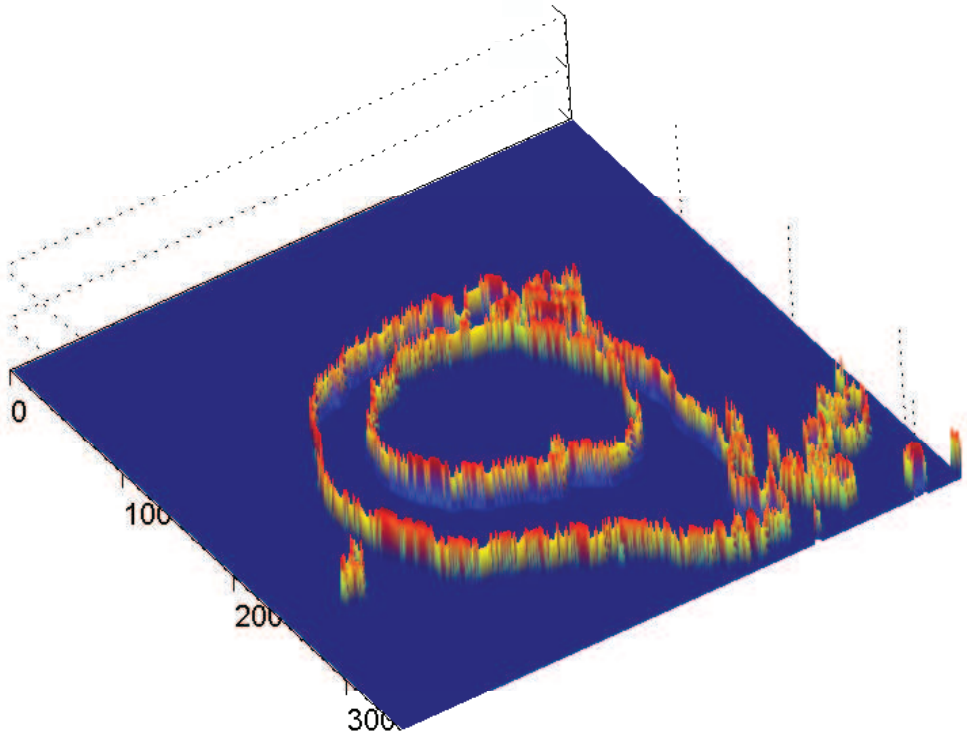
This is a set of approximately parallel lines with slope L. The largest FTLE is approximated by fitting the average line

$$a(t_i) = \frac{1}{\Delta t}\langle \ln d_j(t_i) \rangle,$$

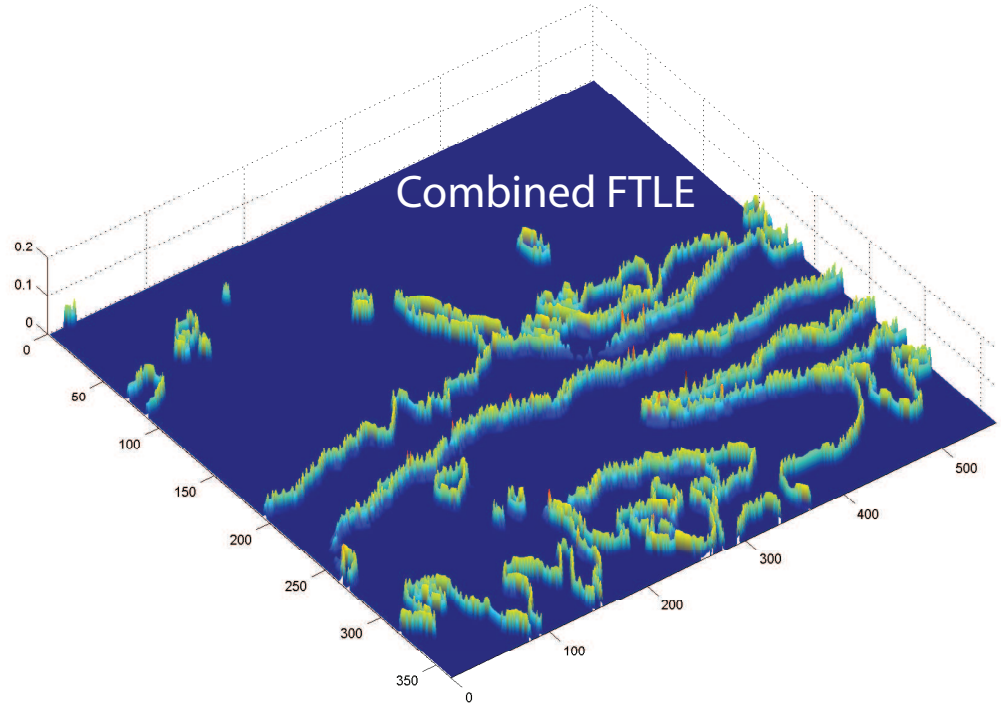
$\langle \cdot \rangle$ denotes the average over j, crucial for small/noisy data sets.

Computed FTLE for a Video Sequence

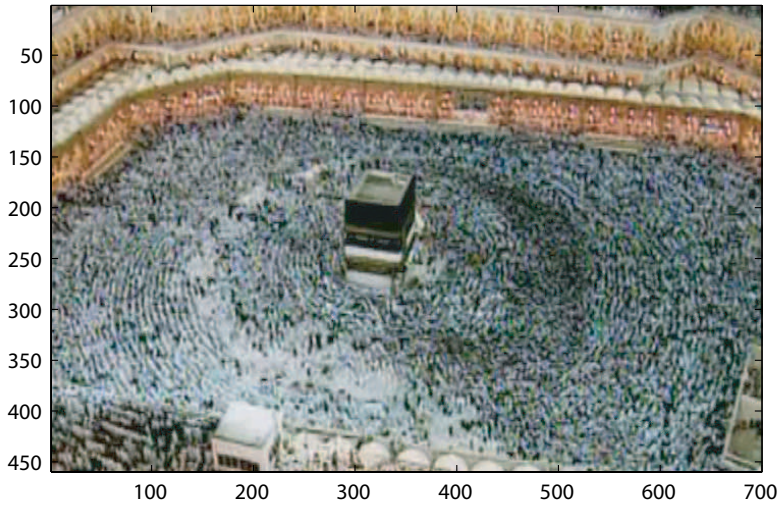


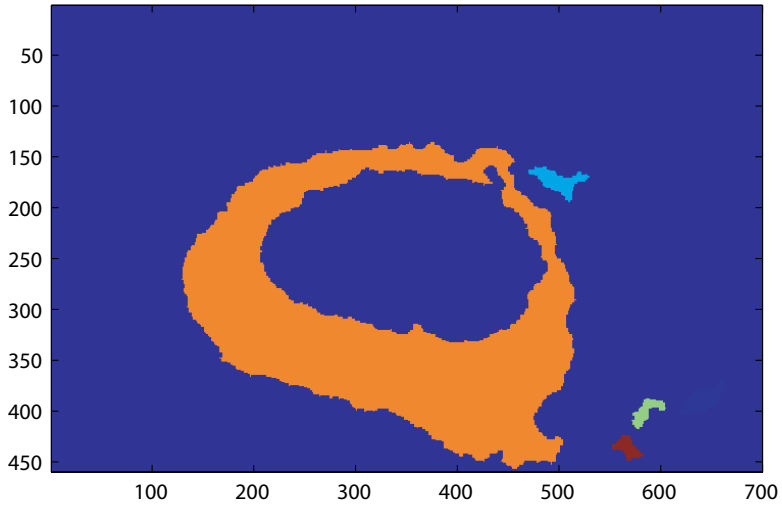


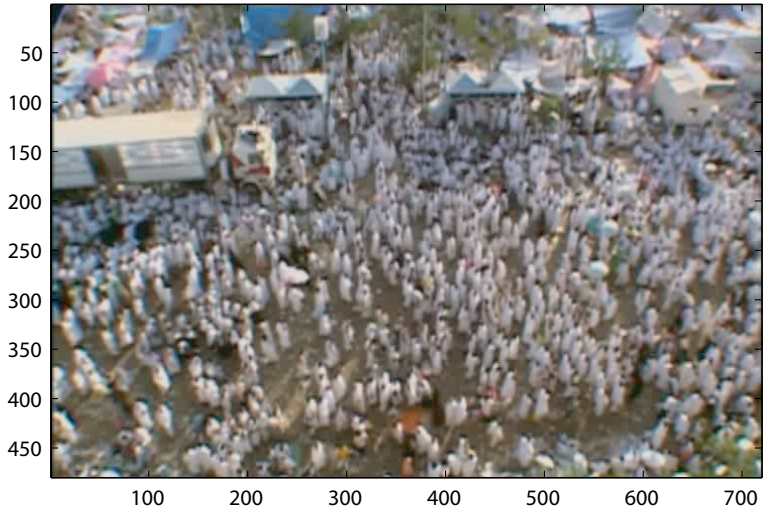
Computed FTLE for a Video Sequence

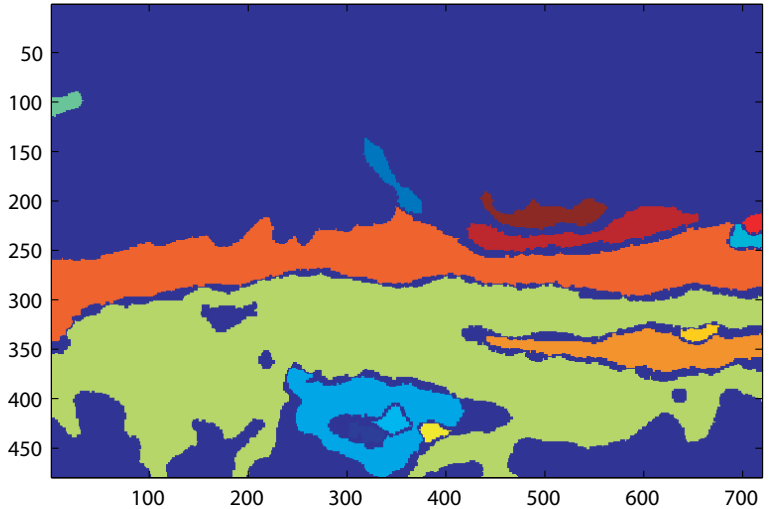


Flow Segmentation for Video Sequences

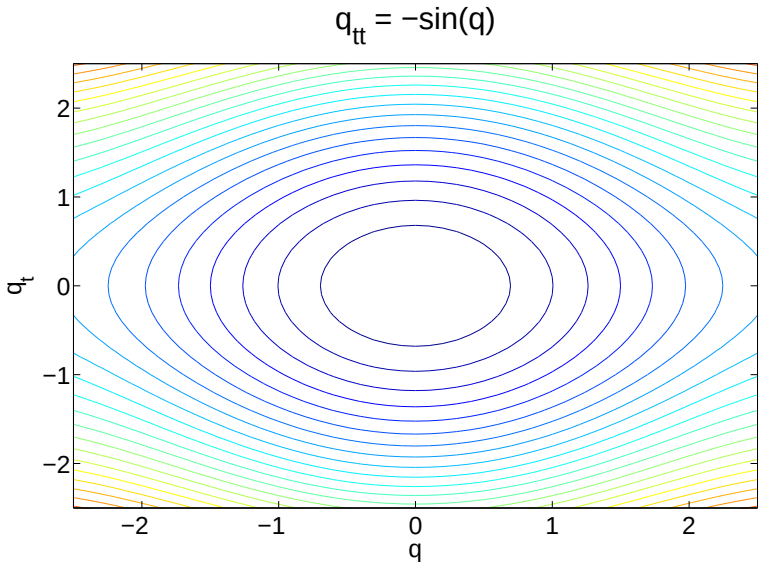


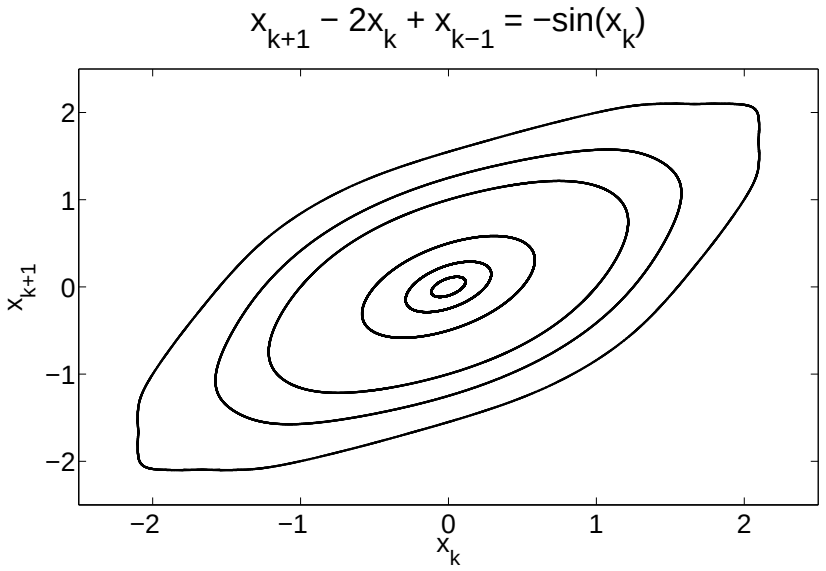






Continuous and Discrete Dynamical Systems: Pendulum Phase Portraits





$$\dot{q} \approx \frac{q(t+h) - q(t)}{h} \qquad \text{and} \qquad \dot{q} \approx \frac{q(t) - q(t-h)}{h}$$

$$\ddot{q} \approx \frac{d}{dt}\frac{q(t+h) - q(t)}{h} \approx \frac{1}{h}\left(\frac{q(t+h) - q(t)}{h} - \frac{q(t) - q(t-h)}{h}\right) = \frac{q(t+h) - 2q(t) + q(t-h)}{h^2}$$

Times Series Data

A time series is a sequence of numbers that represent measurements of some quantity a specific points in time.

For the discrete pendulum problem
$x_{k+1} - 2x_k + x_{k-1} = -\sin(x_k)$ with $x_0 = 1$ and $x_1 = 0$

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \dots \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & -1 & -1.159 & -0.401 & 0.747 & 1.215 & \dots \end{bmatrix} \end{aligned}$$

Phase portrait is obtained by plotting the points (x_{2j-1}, x_{2j}).

In applications, we are given the data but not the rule. We want to extract information about the dynamics using only the data.

Discrete Dynamical Systems

In general, a discrete dynamical system has the form

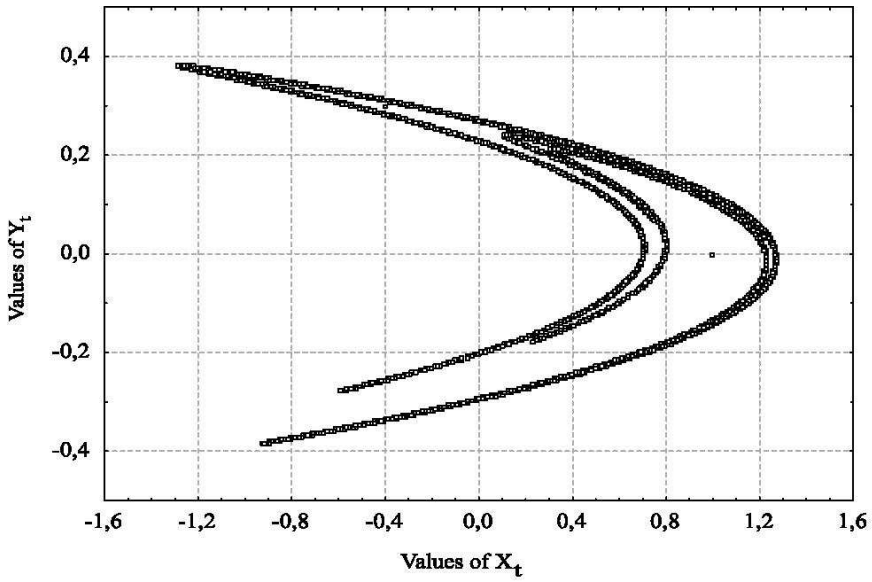
$$\mathbf{z}_{n+1} = \mathbf{F}(\mathbf{z}_n) \quad \text{with specified inital condition} \quad \mathbf{z}_0.$$

Example: The Henon Map $\quad x_{n+1} = 1 + y_n - a x_n^2, \quad y_{n+1} = b x_n$

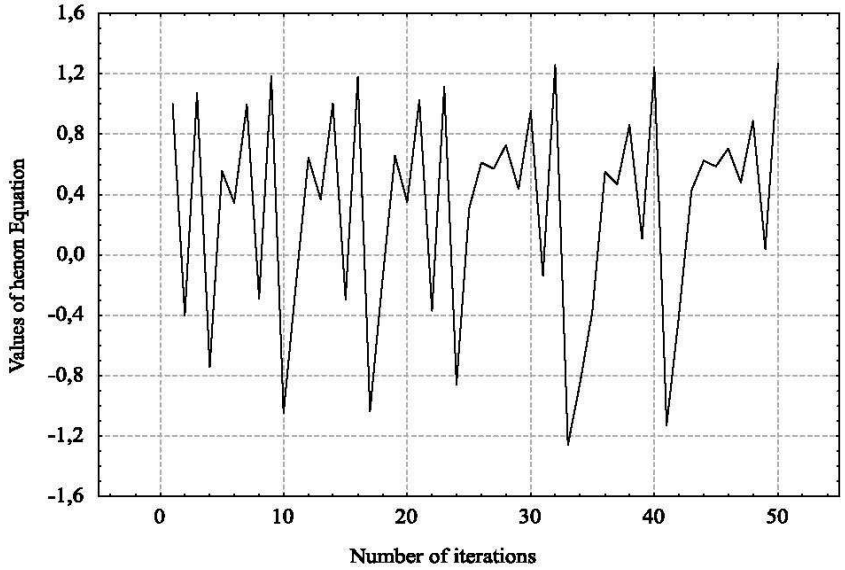
Notice:

- $\mathbf{z}_1 = \mathbf{F}(\mathbf{z}_0), \quad \mathbf{z}_2 = \mathbf{F}(\mathbf{F}(\mathbf{z}_0)), \quad \mathbf{z}_3 = \mathbf{F}(\mathbf{F}(\mathbf{F}(\mathbf{z}_0))), \quad \ldots$
  So it is common to use the notation $\mathbf{F}^k(\mathbf{z}) = \mathbf{F}(\mathbf{F}^{k-1}(\mathbf{z}))$
- The phase space dimension is the dimension of $\mathbf{z}$.
- $\mathbf{z}_*$ is a fixed point if $\mathbf{z}_* = \mathbf{F}(\mathbf{z}_*)$.
- A linearization about the fixed point is $\mathbf{z}_{n+1} = \mathbf{J}(\mathbf{z}_*)\mathbf{z}_n$, where $\mathbf{J}(\mathbf{z}_*)$ is the Jacobian matrix for $\mathbf{F}$ evaluated at $\mathbf{z}_*$.

Henon Map: Time Series and Phase Portrait for $a = 1.4$ and $b = 0.3$



(a)

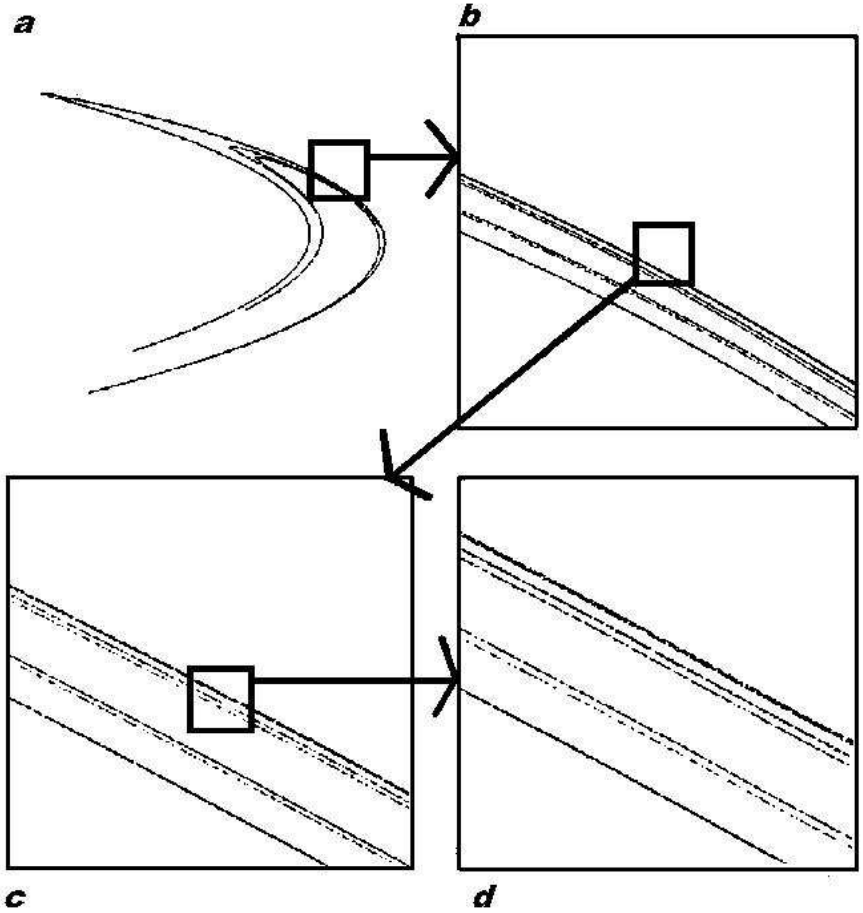


(b)

Note: A time series plot of X_t in (a) shows that the series tends to behave in an aperiodic manner. The phase diagram shows that a "banana-shaped" object appears. This is the Henon attractor, for given starting values of X_t

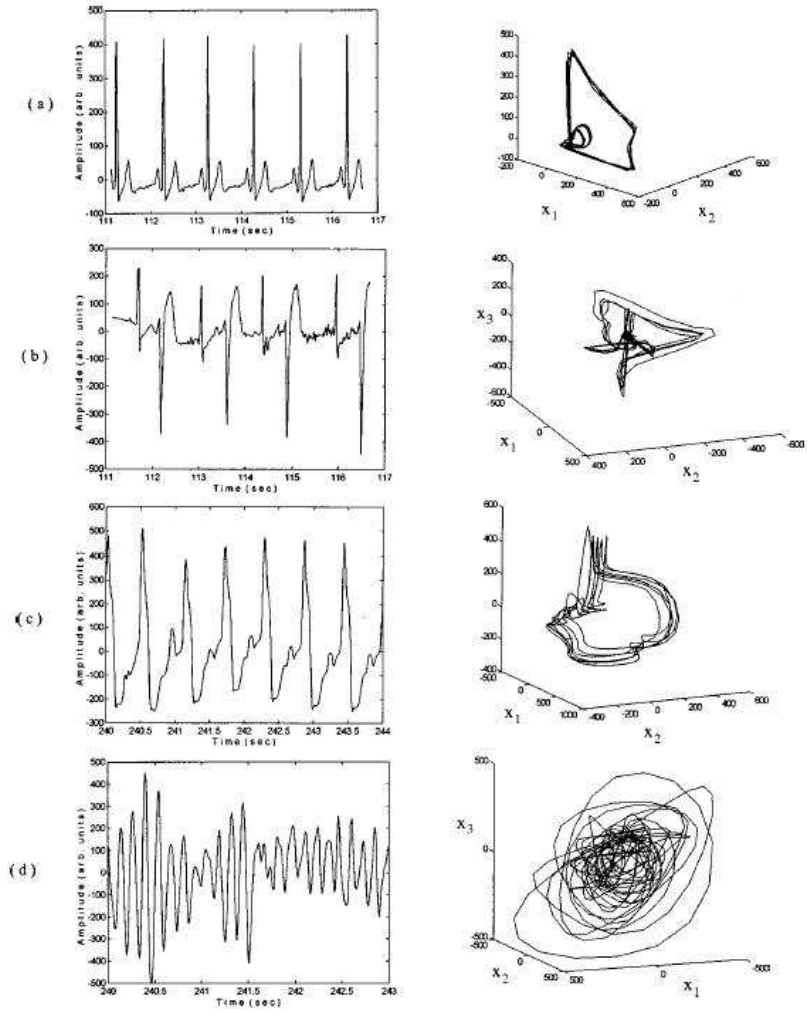
Anthi Katsirikou, Christos Skiadas, Apostolos Apostolou, Giannis Rompogiannakis, (2002) "Virtual libraries: interactive support software and an application in chaotic models", Campus-Wide Information Systems, Vol. 19 Iss: 1, pp.4 - 14

Phase Portrait for the Henon Map



http://www.quatuor.org/english/fractal_33.htm

Time series and Phase Portraits for ECG



Narayana, K.; Govindan, R.B.; Gopinathan, M.S. 1997. Unstable periodc orbits in human cardiac rhythms. Physical Review E. 57(4): 4594-4603.

http://www.sfu.ca/~rpyke/335/projects/francois/report.html

Phase Space Embedding

Given a time series $\mathbf{x} = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & \dots \end{bmatrix}$ we want to find τ and m such that

$$(z_{t-\tau}, z_{t-2\tau}, \dots, z_{t-m\tau}) = \mathbf{F}\left(x_{t-\tau}, x_{t-2\tau}, \dots, x_{t-m\tau}\right).$$

- m is the dimension of phase space
- τ is known as the time delay

The function $\mathbf{F}$ may remain unknown; as long as we have τ and m we can reconstruct phase space and compute important quantities that characterize the dynamics.

How to approximate τ and m will be revealed at a later time.

Discrete Pendulum

As a first order system

$$\dot{q} = p, \qquad \dot{p} = -\sin(q)$$

In discrete form

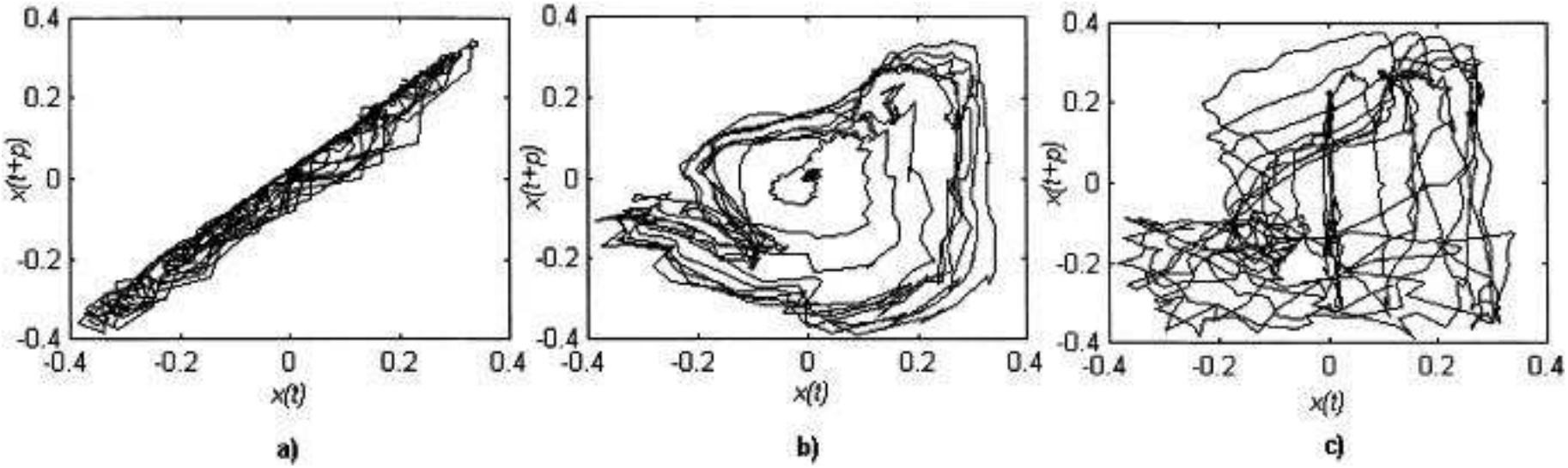
$$q_{n+1} = q_n + hp_n, \qquad p_{n+1} = p_n - h\sin(q_{n+1})$$

In the case of the discrete pendulum we have $\tau = 1$ and $m = 2$

$$(x_{t+1}, x_t) = (z_{t-1}, z_{t-2}) = \mathbf{F}\,(x_{t-1}, x_{t-2})\,.$$

In phase space we plot points (x_{k+1}, x_k), because these correspond to points (q_n, p_n)

Phase Space for Varying Time Delays



(a) time delay is too small

(b) time delay is good

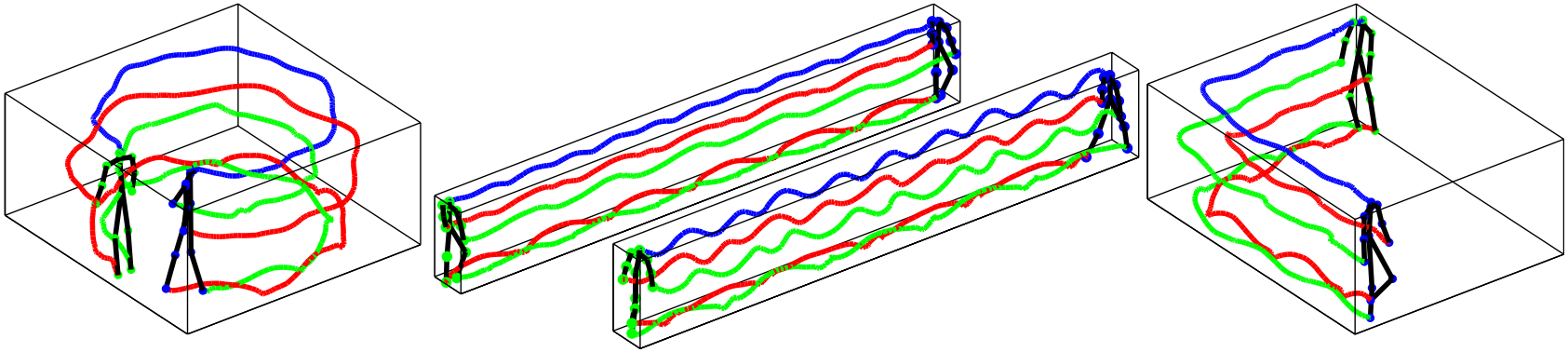
(c) time delay is too excessive

A. Petry and D.A.C. Barone, Speaker identification using nonlinear dynamical features, *Chaos, Solitons & Fractals*, Volume 13, Issue 2, February 2002, Pages 221-231

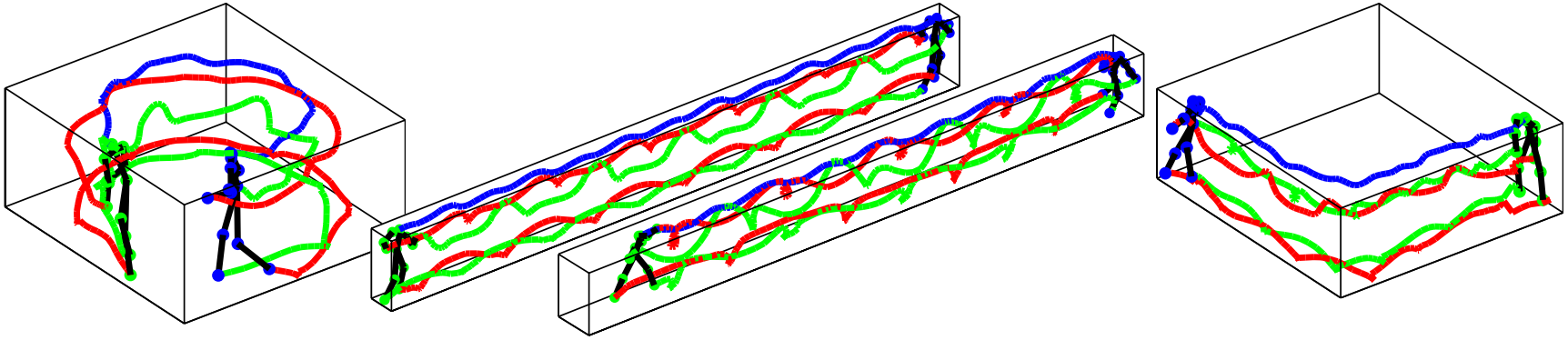
Human Actions

Time Series from Human Motion

Walking

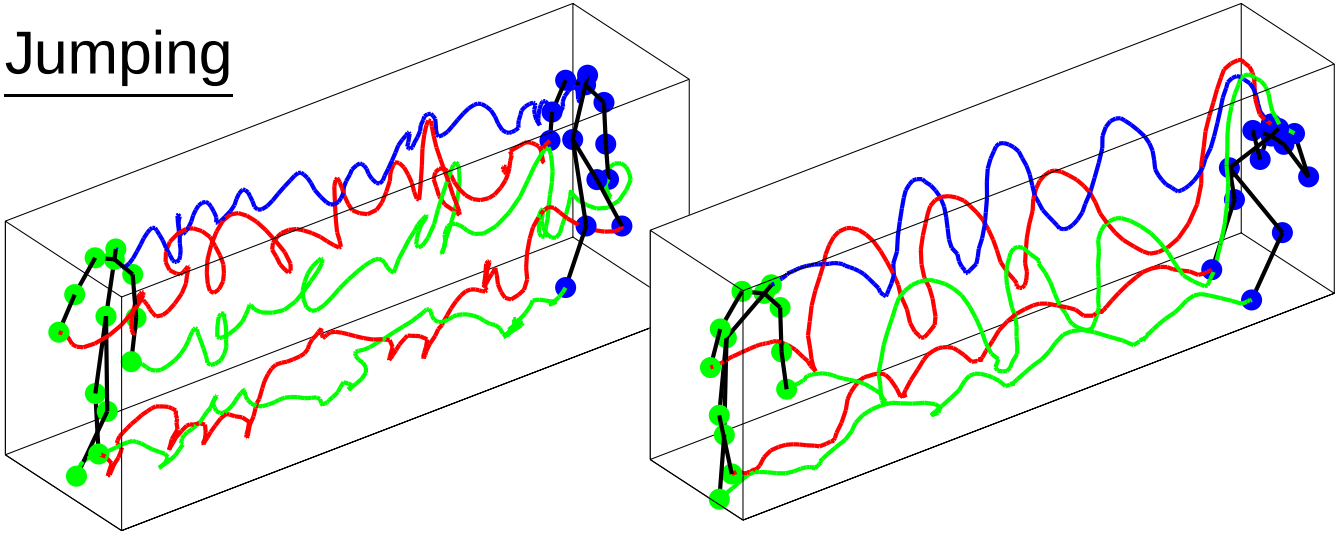


Running

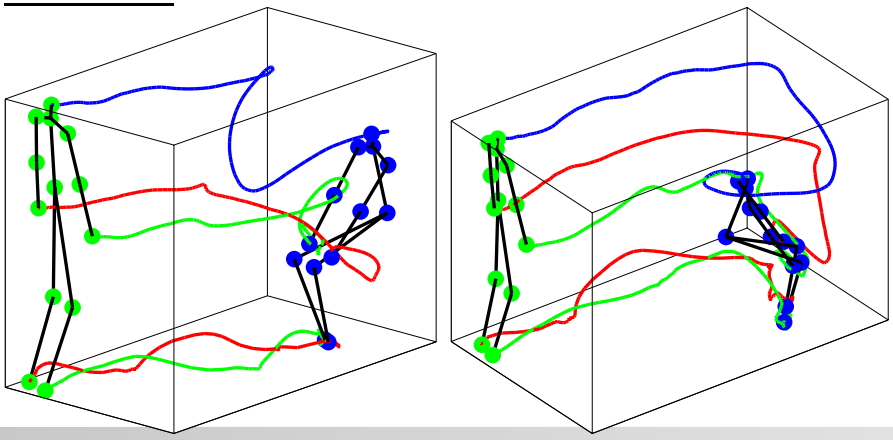


Time Series from Human Motion

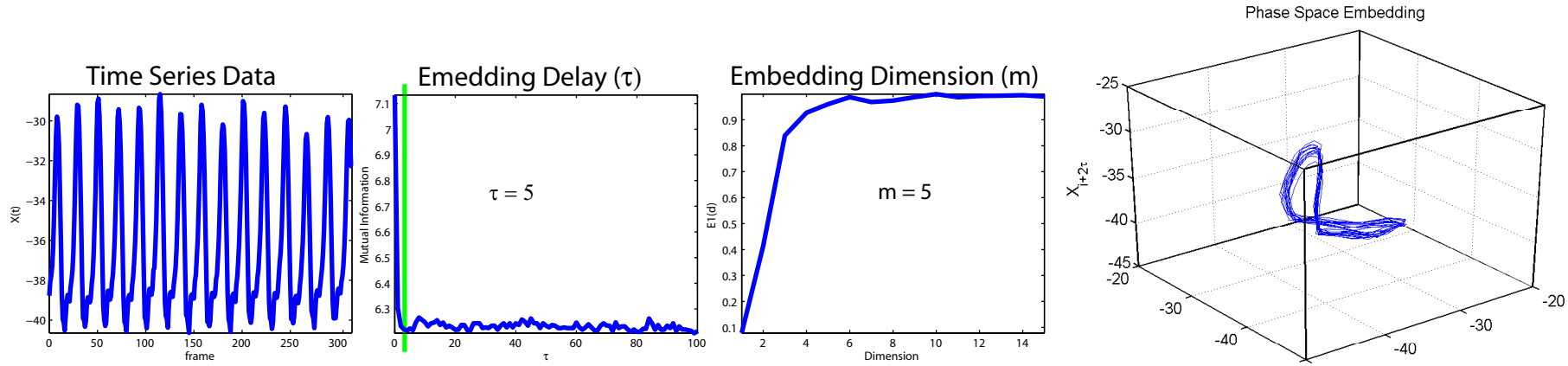
Jumping



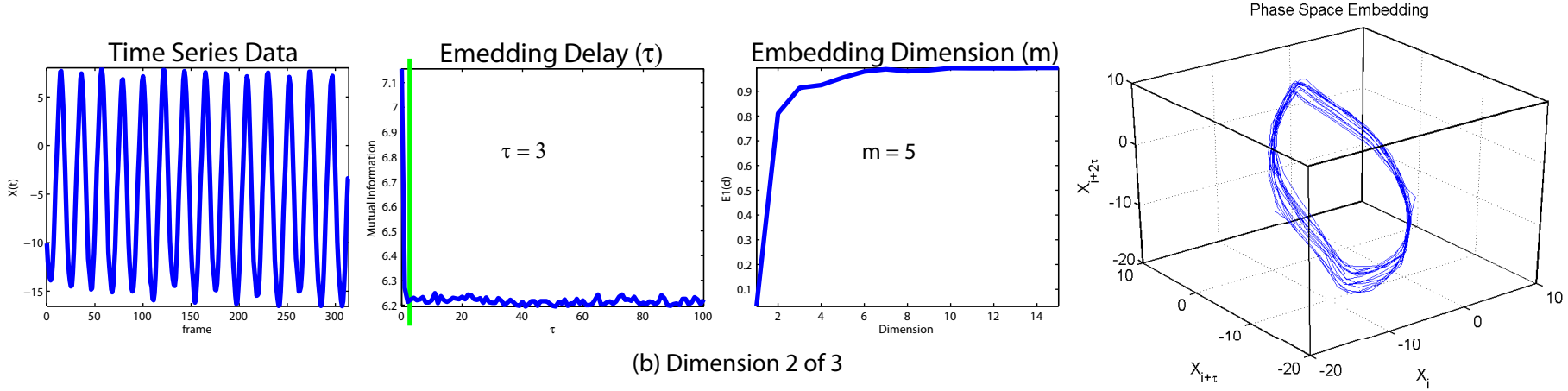
Sitting



From Time Series to Phase Space



(a) Dimension 1 of 3.



(b) Dimension 2 of 3

Recurrence Plots

Given the *delay reconstruction* vectors

$$\mathbf{z}_t = \left(z_{t-\tau}, z_{t-2\tau}, \ldots, z_{t-m\tau}\right),$$

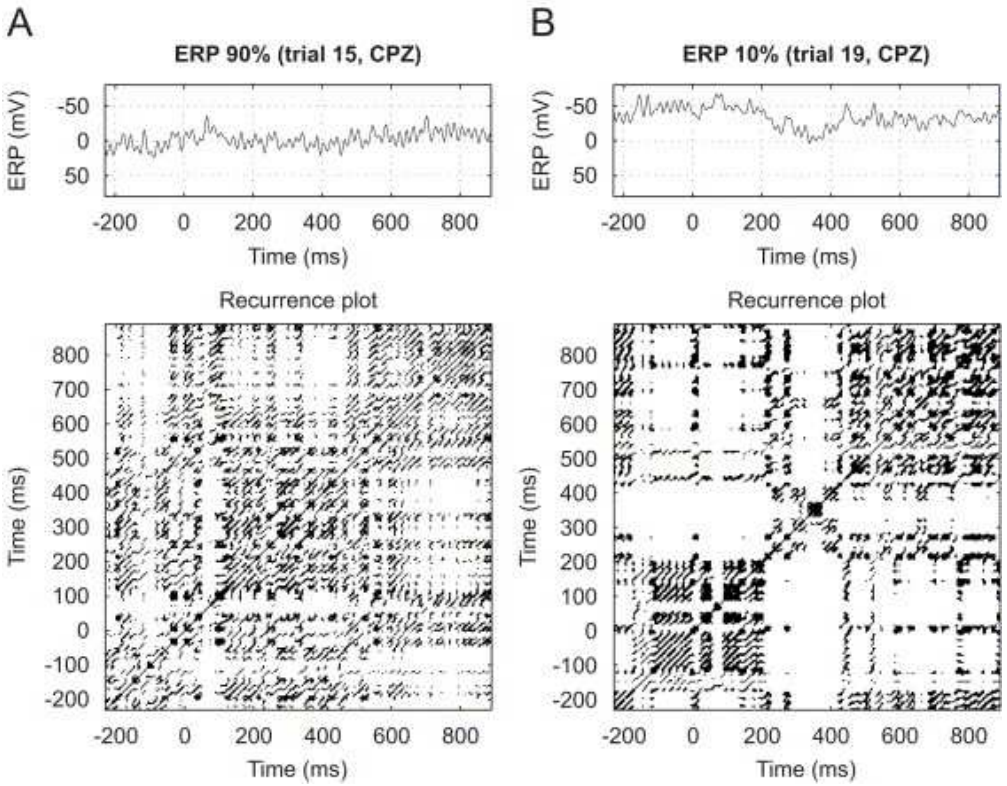
we can visualize recurrence in phase space with

$$M_{i,j} = H(\epsilon - |\mathbf{z}_i - \mathbf{z}_j|).$$

- ϵ is a chosen tolerance.
- H is the Heaviside function $\quad H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

$M_{i,j}$ is 1 if the points $\mathbf{z}_i$ and $\mathbf{z}_j$ are sufficiently close in phase space, 0 otherwise.

Recurrence Plot Examples



N. Marwan, M.C. Romano, M. Thiel, and J. Kurths, Recurrenceplots for the analysis of complex systems, *Physics Reports*, Volume 438, Issues 5-6, January 2007, Pages 237-329

http://www.sciencedirect.com/science/article/pii/S0370157306004066

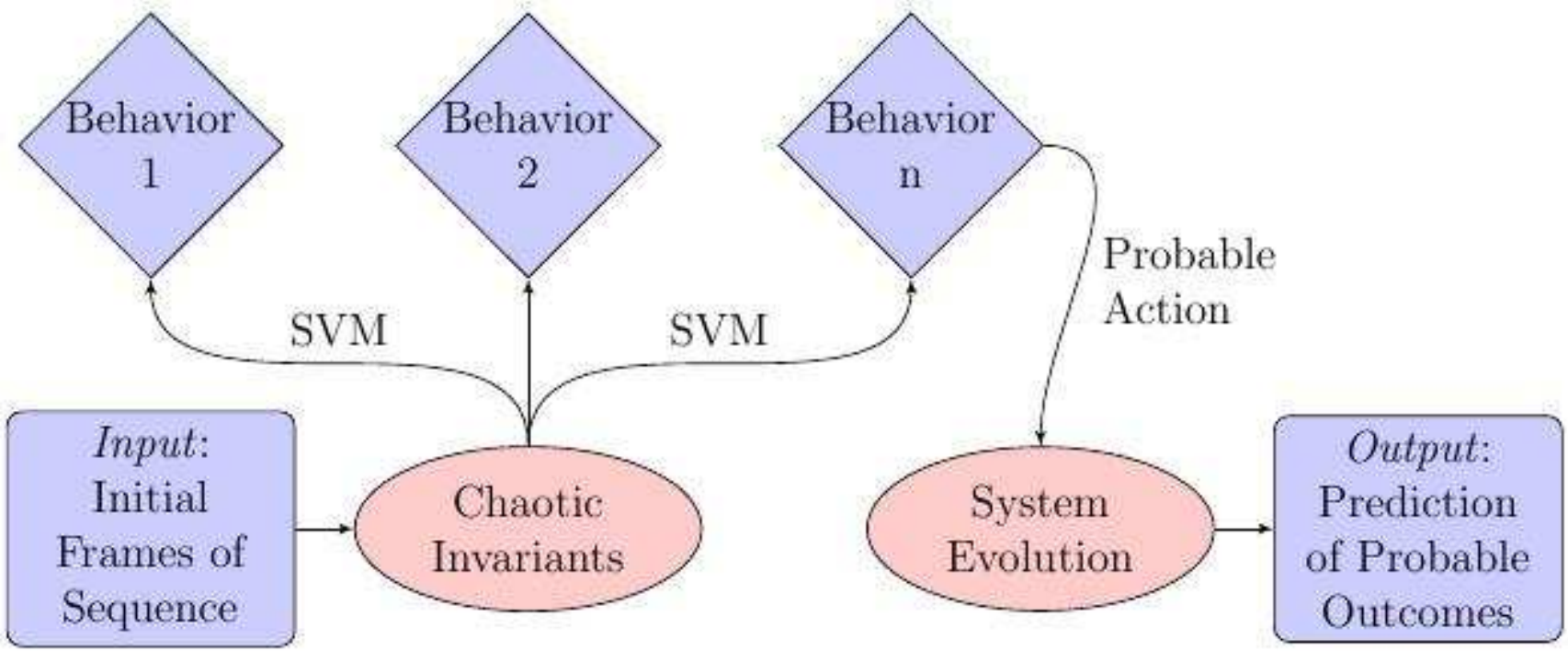
Correlation Integral and Dimension

- Correlation Sum: a measure of spatial organization, giving the mean probability that states at two different times are close.

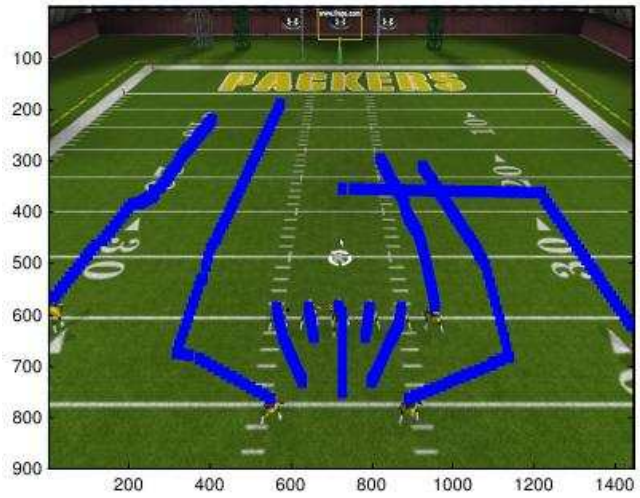
$$C(\epsilon) = \frac{1}{N^2} \sum_{i,j=1}^{N} H(\epsilon - \|\mathbf{z}_i - \mathbf{z}_j\|)$$

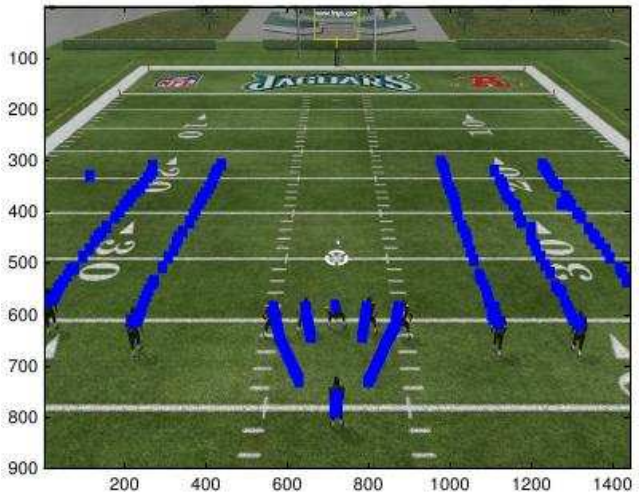
- Correlation Dimension: is effectively a measure of the size of the attractor, denoted by d, and is determined by the relation $C(\epsilon) \approx \epsilon^d$

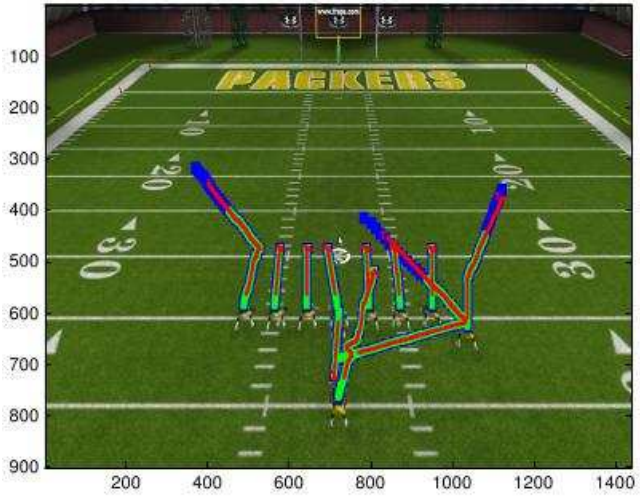
Predicting Outcomes with Chaotic Modeling



Predicting Outcomes in Football Plays







Homework Problems

1. Use Matlab to plot different 2-dimensional phase portraits for the time series that is produced by the Henon map. Use delay times $\tau = 1, 2, \ldots$ and determine which plots give the clearest information about the behavior.
2. Produce a recurrence plot for the Henon map, and compute the correlation sum and the correlation dimension.