Text: Strogatz : Nonlinear dynamics and chaos

Chp 1 : global overview

Definitions

① Dynamical system:

a system of 1 or more variables which evolve in time according to a mathematical rule

We distinguish

- continuous dynamical systems

$$\frac{d\vec{x}}{dt} = F(\vec{x}, t)$$

time is continuous

- discrete dynamical systems

$$\vec{x}(t+\Delta t) = F(\vec{x}(t))$$

or $\vec{x}_{n+1} = F(\vec{x}_n)$

here $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is vector of variables

and F is a matrix operation

Examples.

Continuous

- $\frac{dN}{dt} = -N/\tau_0$
- $\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{cases}$

Discrete

$N_{n+1} = N_n/\tau$

$\begin{cases} a_{n+1} = b_n \\ b_{n+1} = b_n + a_n \end{cases}$

$\Rightarrow$ corresponds to a 2^{nd} order linear differential equation

$$m\frac{d^2h}{dt^2} + b\frac{dh}{dt} + kh = 0$$

can be written in matrix form

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & -b/m \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\rightarrow$ correspond to a 2^{nd} order difference equation

$$a_{n+2} = a_{n+1} + a_n$$

can be written in matrix form

$$\vec{x}_{n+1} = \begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a_n \\ b_n \end{pmatrix}$$

(2) Autonomous / non-autonomous system is one in which the independent variable (t or n) does not / does appear explicitly

$$\begin{cases} \dfrac{dx}{dt} = F(x) & \text{is autonomous} \\ \dfrac{dx}{dt} = F(x,t) & \text{is non-autonomous} \end{cases}$$

Ex :

- $m\dfrac{d^2h}{dt^2} + b\dfrac{dh}{dt} + kh = 0$ is autonomous.

- forced harmonic oscillator

$$m\frac{d^2h}{dt^2} + b\frac{dh}{dt} + kh = A\cos\omega t$$

is not autonomous

We can always transform a non-autonomous system into an autonomous one by adding a "fake" variable.

Ex. let $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_j \end{pmatrix}$ j dimensional

with dynamical system

$$\frac{dx_1}{dt} = f_1(x_1, \ldots x_j, t)$$

$$\vdots$$

$$\frac{dx_j}{dt} = f_j(x_1, \ldots x_j, t)$$

Let's define a new variable $x_{j+1} = t$

then $\frac{dx_{j+1}}{dt} = 1$

so dynamical system becomes

$$\frac{dx_1}{dt} = f_1(x_1, \ldots x_j, x_{j+1})$$

$$\vdots$$

$$\frac{dx_j}{dt} = f_j(x_1, \ldots x_j, x_{j+1})$$

$$\frac{dx_{j+1}}{dt} = 1$$

Exercise: do analogous derivation for a discrete system

(3) <u>Dimension of a dynamical system</u>

Once a dynamical system has been reduced (using method above) to a <u>system</u> of <u>first order</u> <u>autonomous</u> eqns, the dimension of system is equal to the number of dependent variables

<u>Ex.</u> what is dimension of following system? what is it's reduced form?

$$\frac{d^3y}{dt^3} + y^2 t + \frac{d^2y}{dt^2} = \cos t$$

From now on we work only with reduced <u>first order</u> autonomous systems

(4) Linear / nonlinear system

- a linear autonomous system can be expressed as $\frac{d\vec{x}}{dt} = A\vec{x} + B$

$\vec{x}$ = vector
A, B matrices

$$\vec{x}_{n+1} = A\vec{x}_n + B$$

(see examples above)

- A nonlinear system is <u>not</u> linear

<u>Ex.</u> nonlinear pendulum

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0$$

- although nontraditional terminology, forced harmonic oscillator is nonlinear

$$m\frac{d^2h}{dt^2} + b\frac{dh}{dt} + kh = A\cos\omega t$$

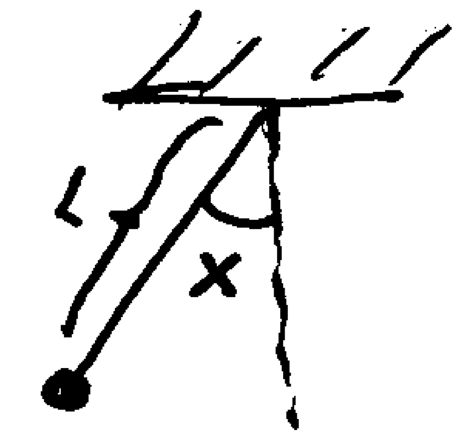
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{1}{m}(-kx_1 - bx_2 + F\cos\omega x_3) \\ \dot{x}_3 &= 1 \end{aligned}$$

- logistic equation

$$x_{n+1} = r x_n (1 - x_n)$$

closer look at the pendulum problem

$$\ddot{x} + \frac{g}{l}\sin x = 0$$



x = ∡ of pendulum from vertical
g = gravity
L = length of pendulum

nonlinear system: $\dot{x}_1 = x_2$
$\dot{x}_2 = -\frac{g}{l}\sin x_1$

- can be solved analytically in terms of elliptic fns
- or if look at small angle approx $\sin x \approx x$ for $x \ll 1$, converts problem to a linear one which can solve

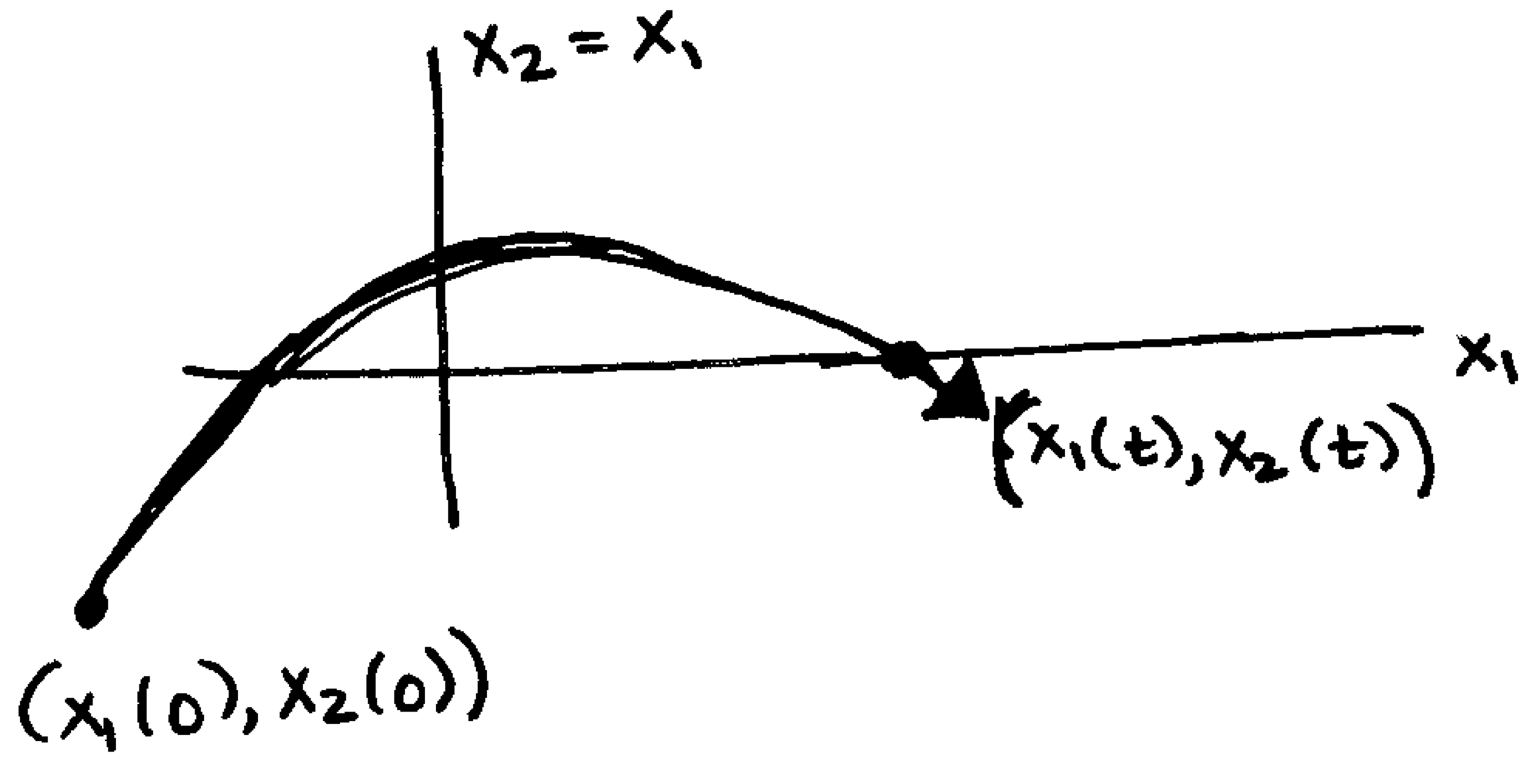
$$\ddot{x} + x = 0$$

but this is very restrictive

- motion of pendulum is simple, at low Energy, it swings back and forth, at high energy, whirls over top.
- Can we extract this information from system without solving explicitly?

Geometric Methods

- suppose know soln to pendulum problem for particular initial condition
- i.e. know $x_1(t)$, $x_2(t)$ which represent position and velocity of pendulum.
- construct phase space with coords (x_1, x_2)



the solution $(x_1(t), x_2(t))$ corresponds to a point moving along a curve in this space.

- This curve is a trajectory. The phase space is completely filled with trajectories, since each pt. can serve as an initial condition.

- Goal: given the system, draw the trajectories and extract information about solutions. In many cases, geometric reasoning allows one to draw trajectories without solving the system.

Note: for general system, the phase space is space with coords $x_1, x_2, \ldots x_n$.
n-dimensional system

Flows on the line : 1-D dynamical systems

We consider systems of the form

$$\frac{dx}{dt} = \dot{x} = F(x)$$

with
- $F(x)$ a smooth function
- x a real-valued variable dependent on time
- Note that f is <u>not</u> explicitly time dependent

For these systems, there exists a closed form solution:

$$\frac{dx}{F(x)} = dt$$

$$\int \frac{dx}{F(x)} = \int dt = t$$

Problem:
- Can we evaluate this integral?
- If we can't, can we still say something about the system?

Ex. easy solvable system

$$\dot{x} = x^2 \qquad x(0) = 1$$

$$\frac{dx}{x^2} = dt$$

$$x = \frac{-1}{t+c}$$

$$x(0) = 1 = \frac{-1}{c} \Rightarrow c = -1$$

$$x = \frac{1}{1-t}$$

- what do we do when integration is hard?

Ex. $\dot{x} = \sin x \qquad \Rightarrow \quad t = \int \frac{dx}{\sin x}$

$x(0) = x_0$

$$t = \ln\left(\frac{\operatorname{cosec} x_0 + \cot x_0}{\operatorname{cosec} x + \cot x}\right)$$

- solution is difficult to interpret so can we obtain a qualitative handle on the behavior of the solution?

<u>use graphical method</u>

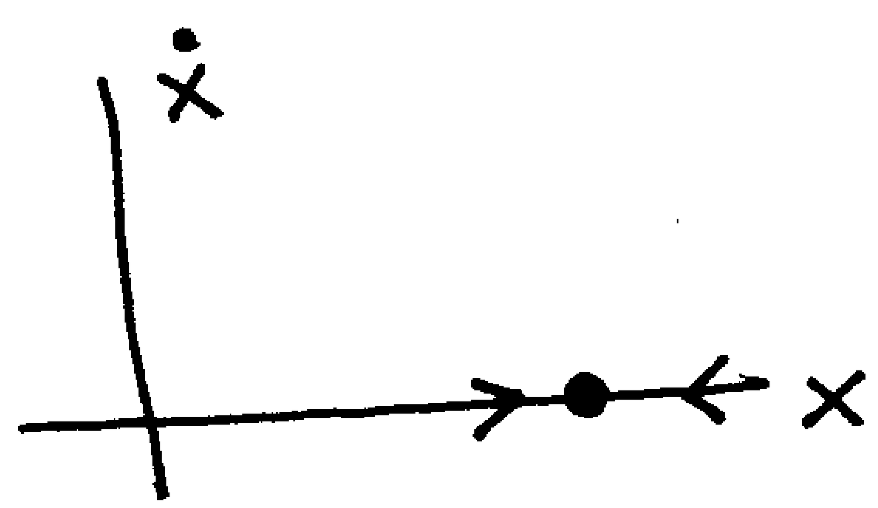
Question: For an arbitrary x_0, what happens as $t \to \infty$

Graphical analysis:

- think of t as time
  x as position of imaginary particle on real line
  $\dot{x}$ as velocity of the particle
- the differential eqn $\dot{x} = \sin x$ represents a vector field on the line
  - it dictates the velocity vector $\dot{x}$ at each x

to **sketch vector field**

Plot



draw arrows on the x axis to indicate the corresponding velocity vector at each x

Physical picture

- Think of a fluid flowing along the x axis w velocity that varies from place to place as $\dot{x} = \sin x$
  - Flow is to the right when $\dot{x} > 0$
  - Flow is to the left when $\dot{x} < 0$
  - Flow is 0 when $\dot{x} = 0 \Rightarrow$ fixed points.

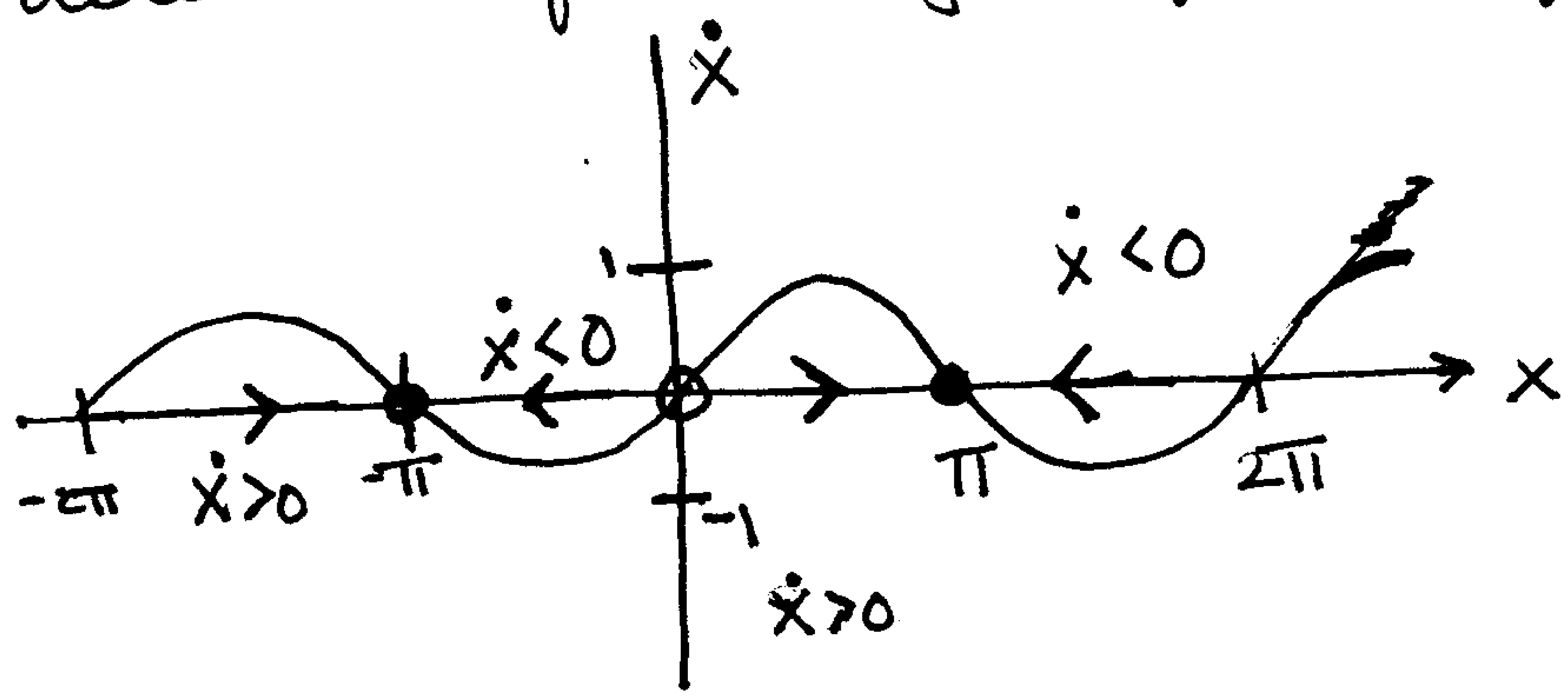
notation

● ≡ stable fixed point (or "attractor" or "sink")

o ≡ unstable fixed point (or "repellers" or "sources")

Procedure

- draw $\dot{x}$ as fn of x
- identify fixed pts $\dot{x} = 0$
- determine from sign of $\dot{x}$ the flow direction



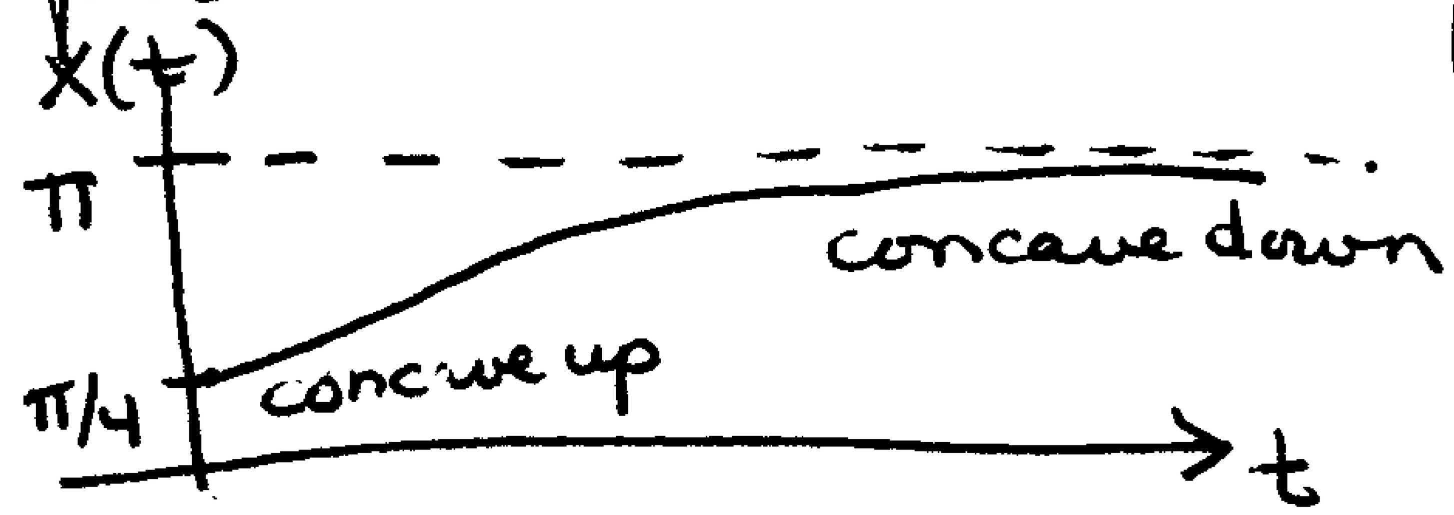
$\dot{x} = \sin x$

$x = \ldots, -\pi, \pi, \ldots (2k+1)\pi$ stable fixed pts

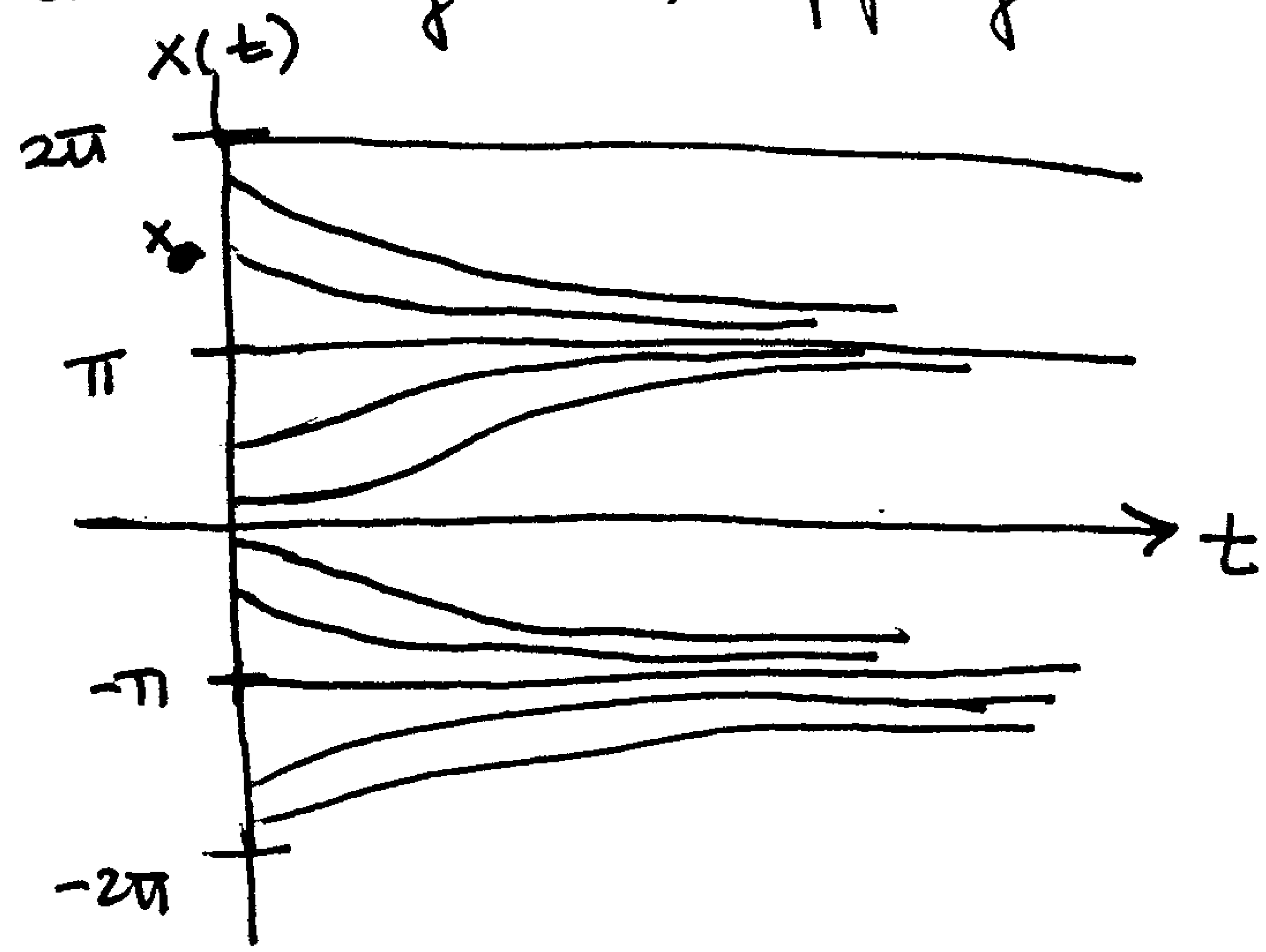
$x = \ldots, -2\pi, 0, 2\pi, \ldots 2k\pi$ unstable fixed pts

If $x_0 = \pi/4$, what is behavior as $t \to \infty$?

- particle starting at $x_0 = \pi/4$ moves to right faster & faster until it crosses $x = \pi/2$ (where $\sin x$ has max). The particle then starts slowing down and eventually approaches the stable fixed pt at $x = \pi$

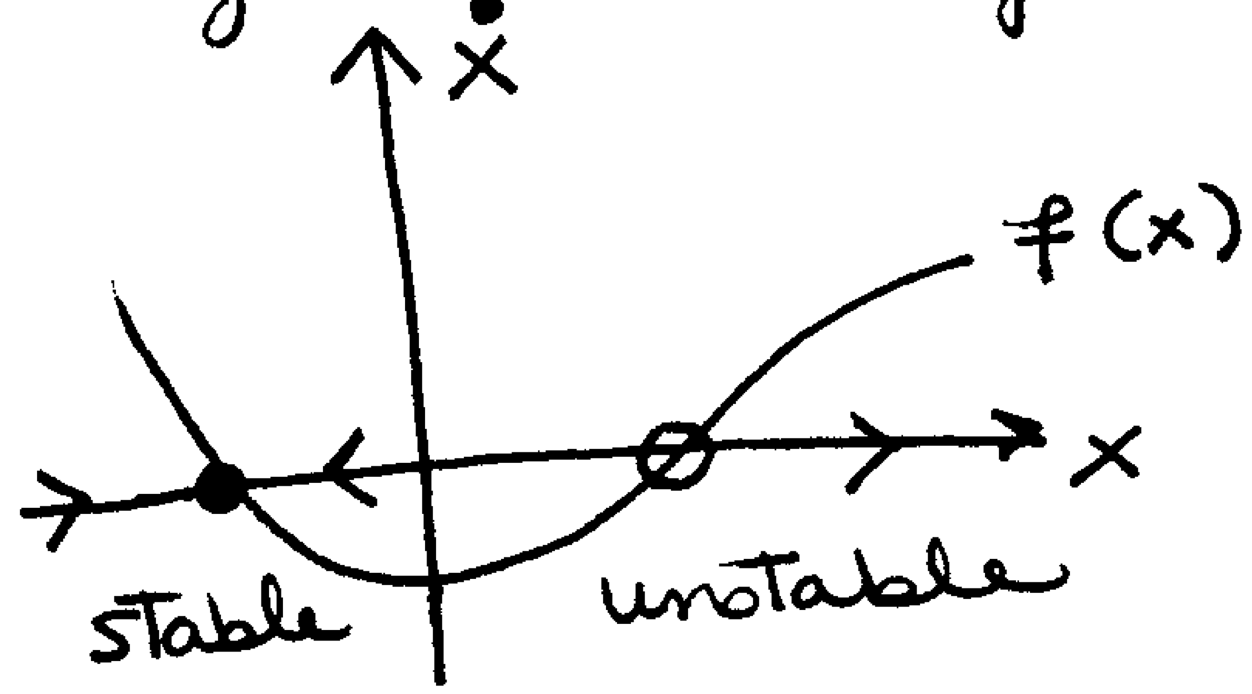


For arbitrary x_0, apply similar reasoning



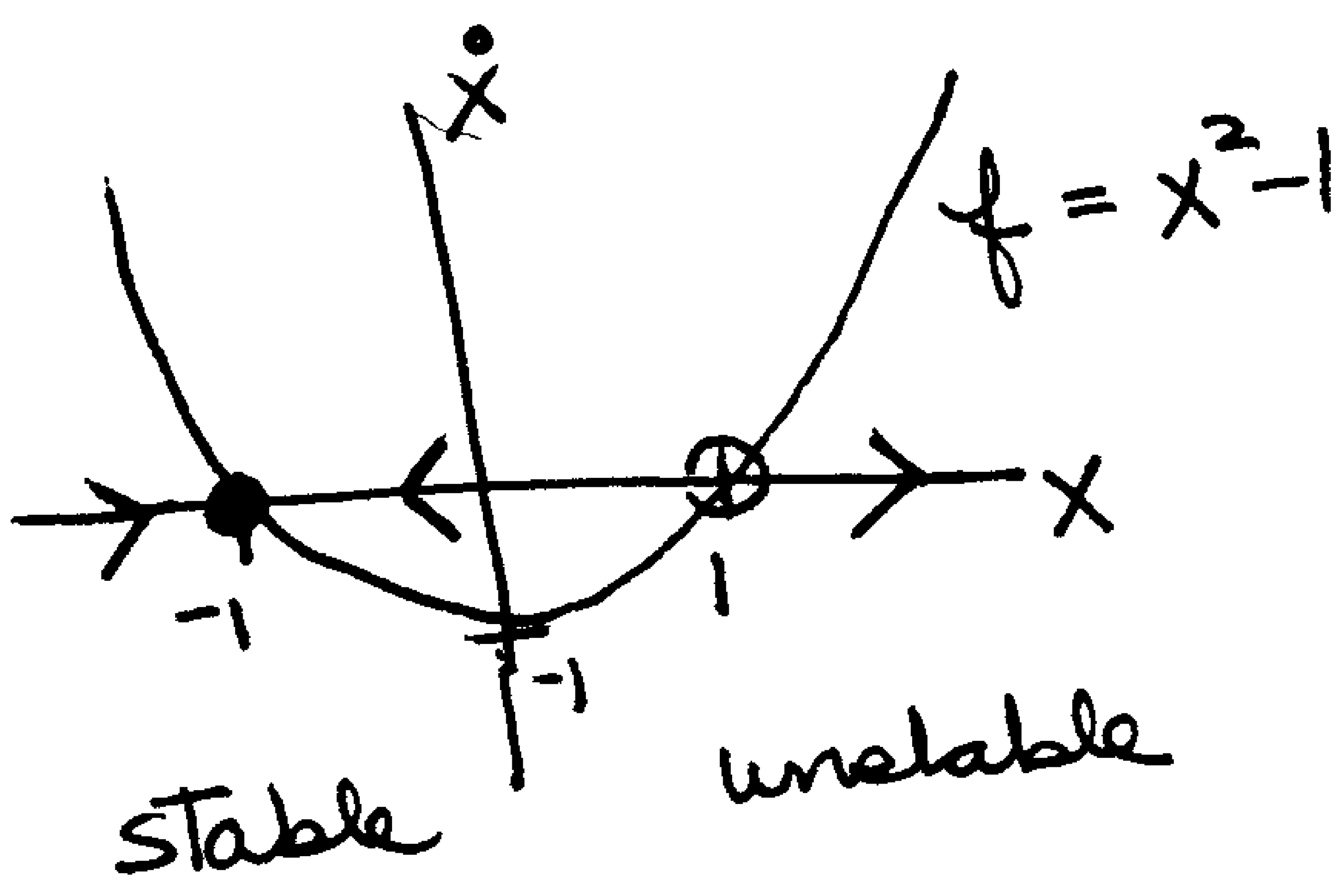
Fixed Points and Stability

For a general 1-D system $\dot{x} = f(x)$



- an imaginary fluid flowing along the real line x (*phase space*)
- to find solution to $\dot{x} = f(x)$, $x(0) = x_0$ place imaginary particle (*phase pt*) at x_0 and watch how carried by fluid flow
- as particle moves along x axis, it's position as function of time given by $x(t)$ (*trajectory*)
- Picture showing all qualitatively different trajectories of system is *phase portrait*
- Appearance of phase portrait controlled by fixed pts x^* defined by $f(x^*) = 0$
- fixed ~~pts~~ represent (stable or unstable) equilibrium solutions of $\dot{x} = f(x)$

Ex. Find all fixed pts for $\dot{x} = x^2 - 1$ and classify stability



- fixed points $f(x) = 0$
  $\Rightarrow \quad x = \pm 1$

- stability obtained from vector field sketch

- <u>Note</u>: definition of stable equilibrium is based on small disturbances. Hence $x = -1$ is <u>locally</u> but not <u>globally</u> stable