Oct. 18th

Linear Stability Analysis

- so far relied on graphical methods to determine stability of fixed points.
- Want a quantitative measure of stability, such as the rate of decay to a stable fixed point

This can be obtained by linearizing about the fixed point x^*

Let $\eta(t) = x(t) - x^*$ be a small perturbation from the fixed point x^*

then $\dot{\eta}(t) = \frac{dx}{dt}$ since x^* is a constant.

- Hence $\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$

$$= f(x^*) + \eta f'(x^*) + O(\eta^2)$$ by Taylor's expansion

Since x^* is a fixed pt, $f(x^*) = 0$

$$\dot{\eta} = \eta f'(x^*) + O(\eta^2)$$

- Now if $f'(x^*) \neq 0$, the $O(\eta^2)$ term is negligible and thus

$$\dot{\eta} \simeq \eta f'(x^*) \qquad \text{(a linear ODE)}$$

- This is the linearization about x^*.
- The slope $f'(x^*)$ at the fixed point determines its stability
  - the perturbation $\eta(t)$ grows exp if $f'(x^*) > 0$
    decays exp if $f'(x^*) < 0$
  - if $f'(x^*) = 0$, the $O(\eta^2)$ terms are not negligible for stability and need to do a nonlinear analysis

<u>Ex.</u> $\dot{x} = \sin x$

- fixed pts occur at $f(x) = \sin x = 0 \Rightarrow x^* = k\pi$
  $k = 0, \pm 1, \pm 2, \ldots$
- $f'(x^*) = \cos k\pi = \begin{cases} +1 & k \text{ even; unstable} \\ -1 & k \text{ odd; stable} \end{cases}$

Ex. $\dot{N} = rN(1 - N/k) = f(N)$

- fixed pts occur $\quad f(N) = rN(1-N/k) = 0$

$$\Rightarrow N^* = 0, k$$

- $f'(N^*) = r(1-N/k) + rN(-1/k)$

$$= r - 2rN/k$$

$$\Rightarrow f'(0) = r \quad , \quad f'(k) = -r$$

- $f'(N) = \begin{cases} r & N = 0 \text{ unstable} \\ -r & N = k \text{ stable} \end{cases}$

- We now have a measure of how stable a fixed pt is – determined by the magnitude of $f'(N^*)$

- the magnitude $|f'(N^*)|$ provides the growth or decay rate

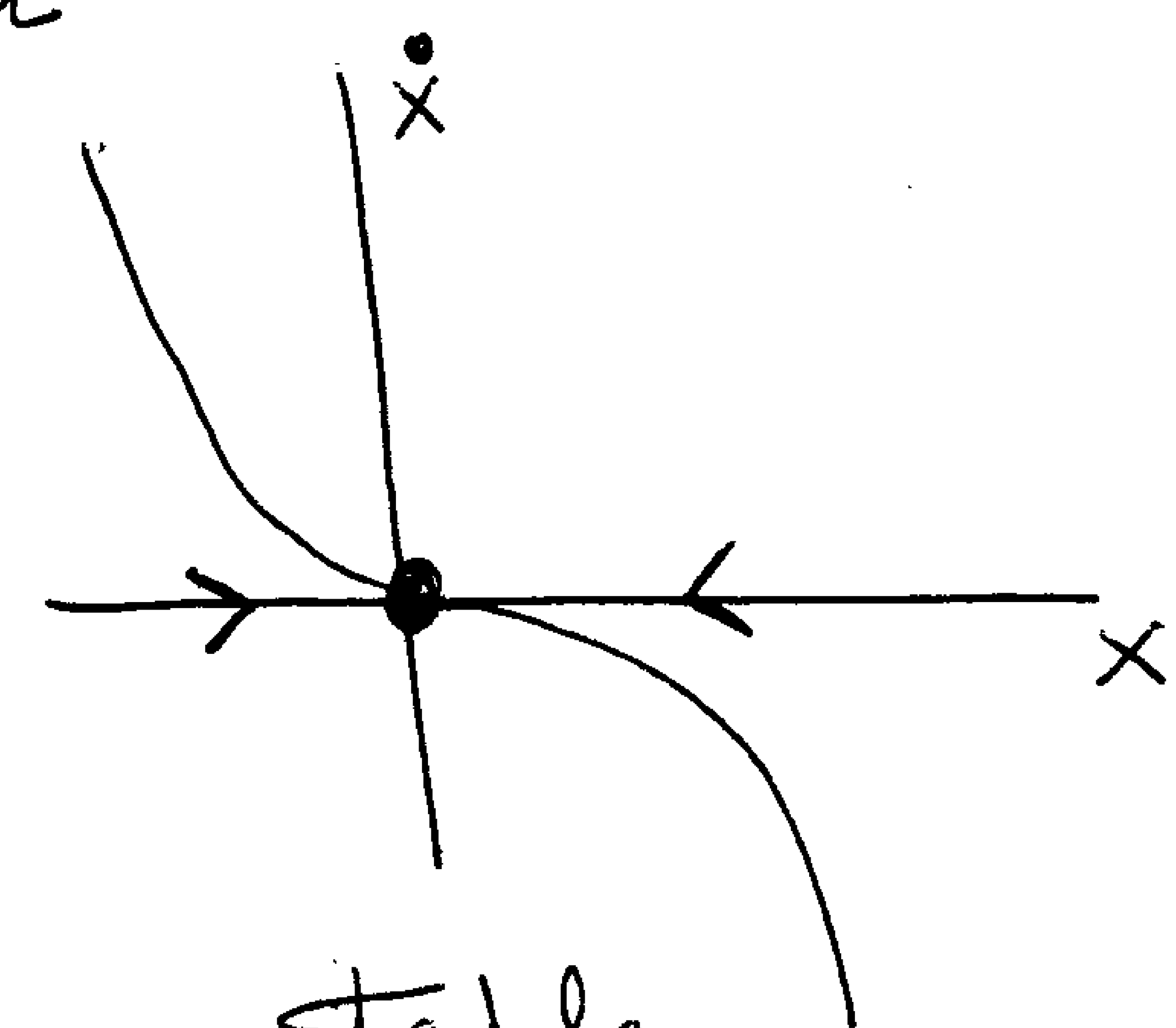
- Characteristic time scale is $|f'(N^*)|^{-1} = 1/r$

time required for $N(t)$ to vary significantly in nghbhd of x^*

Ex Case where $f'(x^*) = 0$

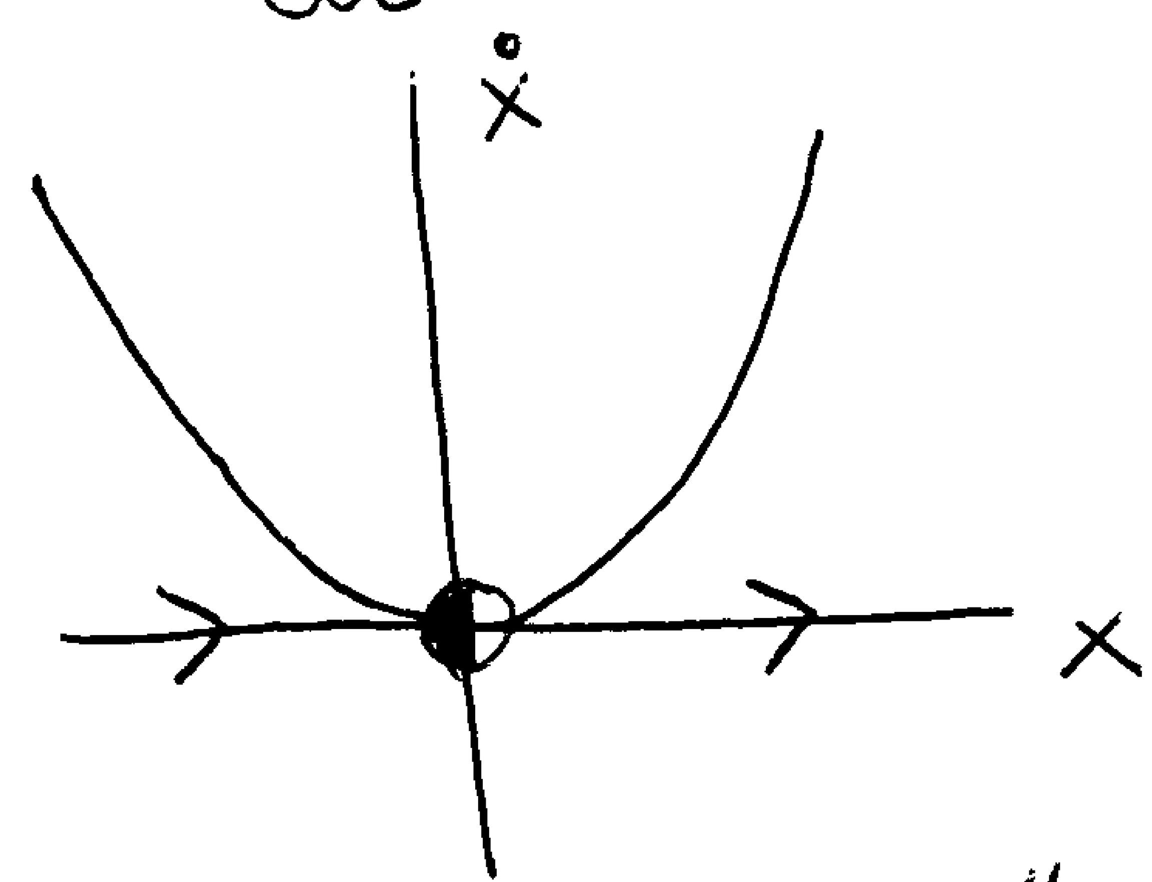
- Nothing can be said in general.
- We can use graphical methods to determine stability on case-by-case basis

a) $\frac{dx}{dt} = -x^3$



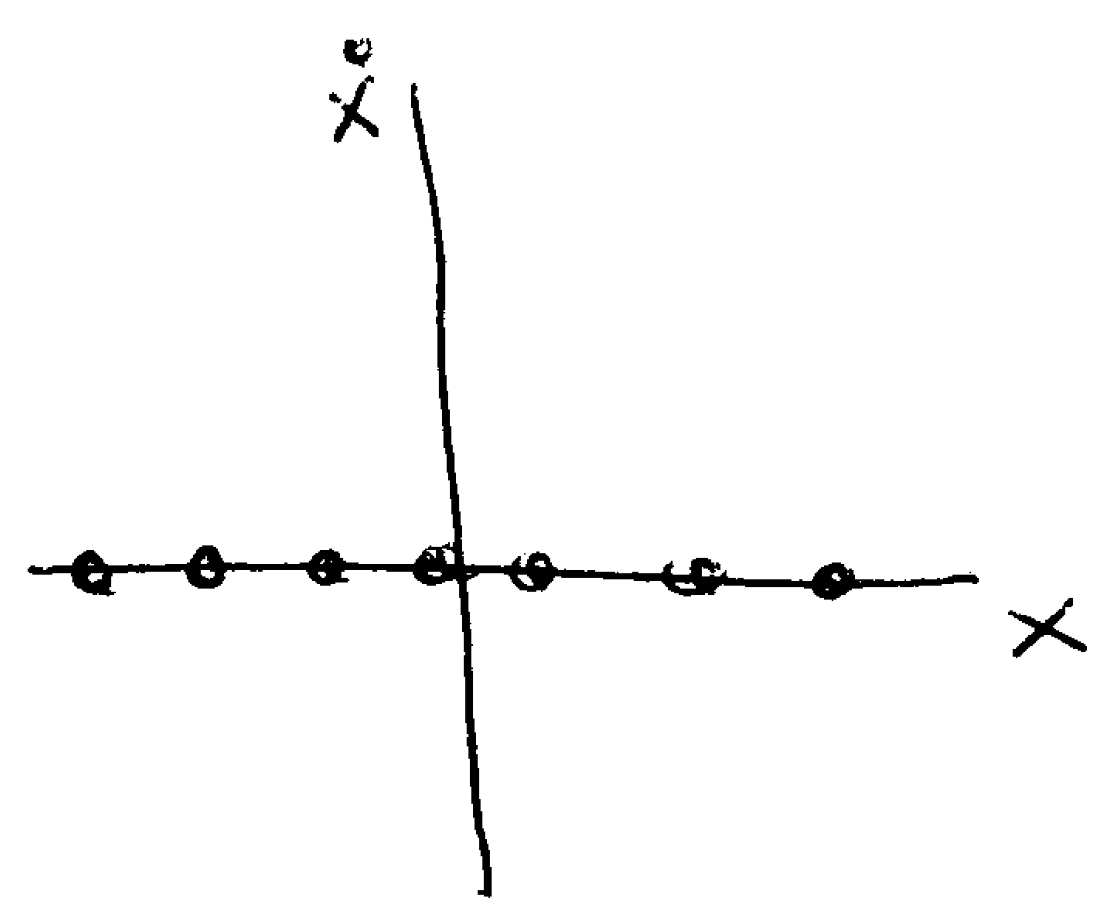
stable

b) $\frac{dx}{dt} = x^2$



"half stable"

c) $\frac{dx}{dt} = 0$.



a whole line of fixed points; perturbations neither grow or decay

Existence and Uniqueness of $\dot{x} = f(x)$ solutions

- need to be aware of pathological cases

Ex. $\dot{x} = x^{1/3}$ $x(0)=0$

- fixed points $f(x)=0 \Rightarrow x^* = 0$

  so $x(t)=0$ is a solution

- another solution;

$$\int x^{-1/3}dx = \int dt$$

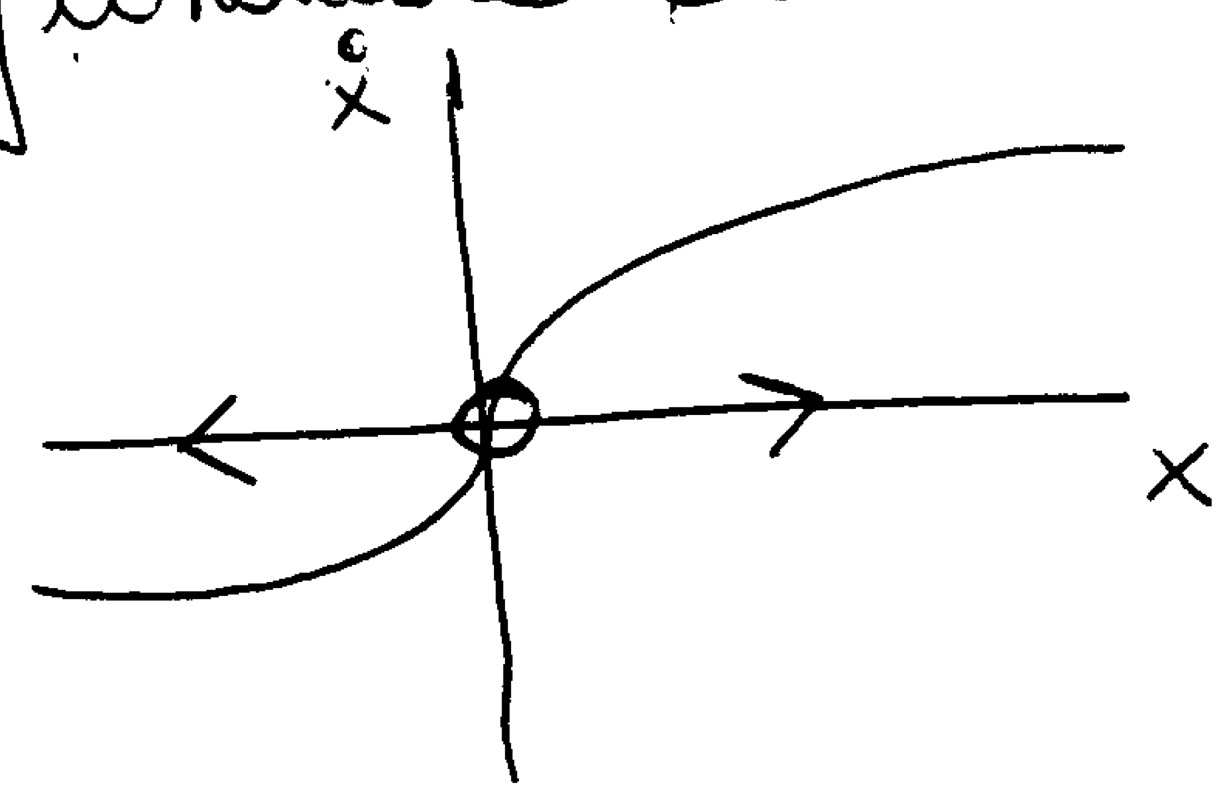
$$\frac{3}{2}x^{2/3} = t + c$$

$$x(0)=0 \Rightarrow c = 0$$

$$x(t) = \left(\frac{2}{3}t\right)^{3/2}$$

- actually infinitely many solutions

- the vector field shows that $x^* = 0$ is very unstable since $f'(0) \to \infty$



Existence and Uniqueness Theorem

- Consider $\dot{x} = f(x)$, $x(0) = 0$

- Suppose $f(x)$, $f'(x)$ are continuous on an open interval R of the x-axis and $x_0 \in R$.

- Then the initial value problem has a solution $x(t)$ on some interval $(-\tau, \tau)$ about $t=0$ and the <u>solution is unique</u>.

★ If $f(x)$ is smooth enough, then solutions exist and are unique

Ex.

$\dot{x} = 1 + x^2 = f(x) \qquad x(0) = x_0$

- Discuss existence and uniqueness
- Do Solutions exist for all time.

— $f(x)$, $f'(x)$ are continuous for all x, so solutions exist and are unique for any x_0

— **But** the theorem does not say they exist for all time !

If $x_0 = 0$, $\dfrac{dx}{dt} = 1 + x^2 \Rightarrow \dfrac{dx}{1+x^2} = dt$

$\tan^{-1} x = t + c$

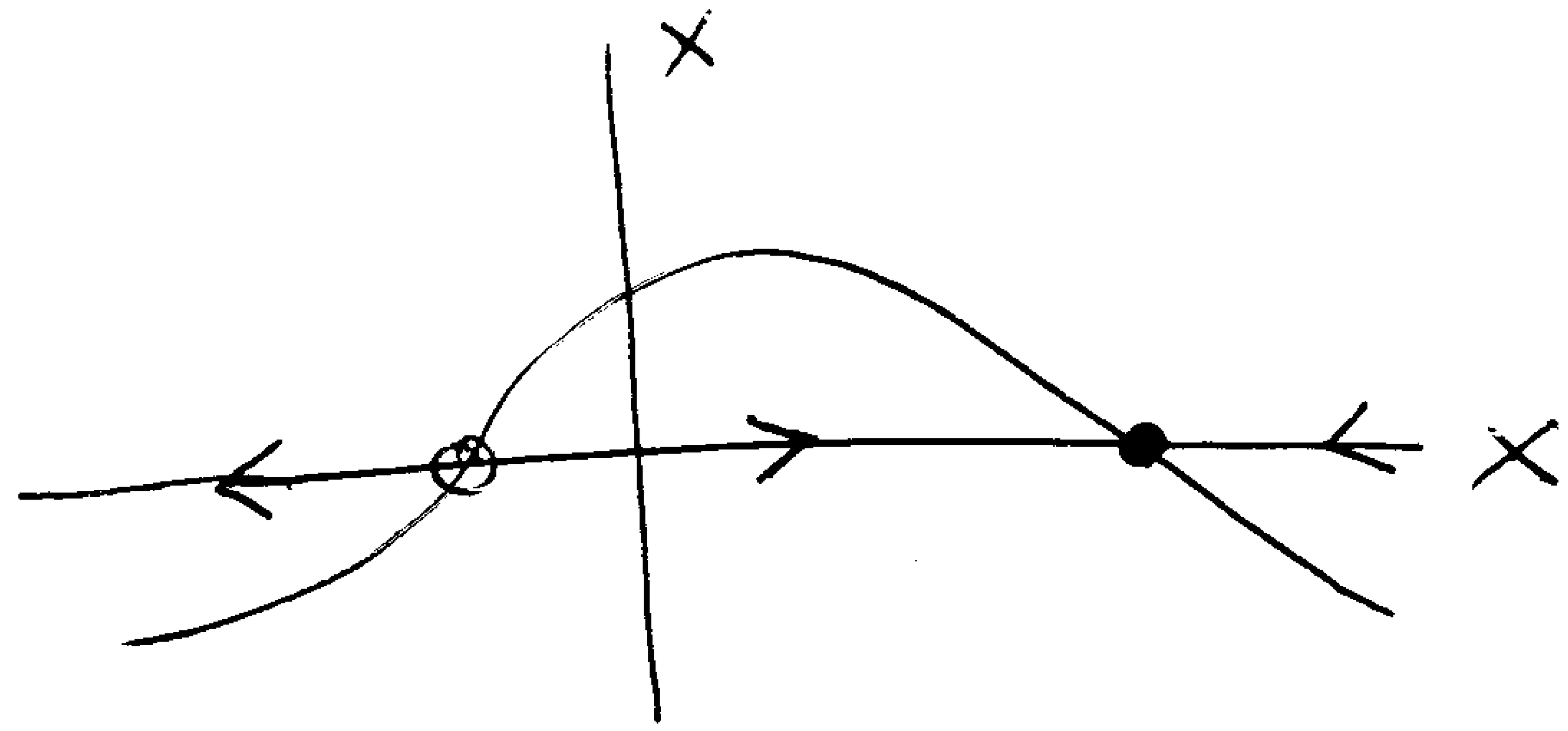
$\tan^{-1}(0) = 0 = c \quad \Rightarrow x = \tan t$

- this solution exists only for $-\frac{\pi}{2} < t < \pi/2$ since $x(t) \to \pm\infty$ as $t \to \pm\pi/2$
- Thus for $|t| > \pi/2$, no solution exists given $x_0 = 0$

The Impossibility of Oscillations

- Fixed pts dominate the dynamics of first order systems
- In all our ex's, all trajectories either approach a fixed pt or diverge to $\pm\infty$.

★ these are only things that can happen for a vector field on the line; trajectories increase or decrease monotonically, or remain constant.



★ There are no periodic solutions to $\dot{x} = f(x)$ since $\dot{x} = f(x)$ corresponds to flow on a line

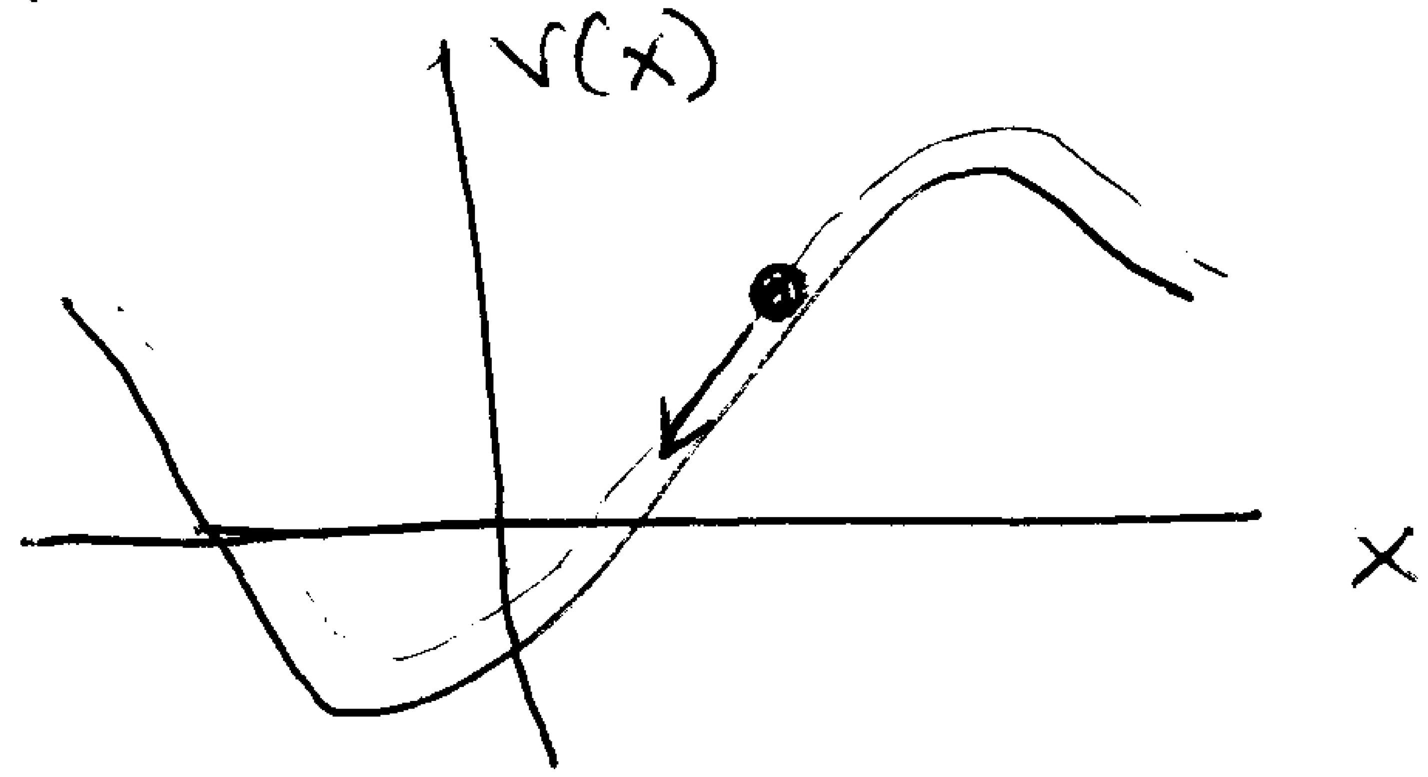
The phase point never reverses direction

Potentials

- Another way to visualize the dynamics of $\dot{x} = f(x)$, by analogy with classical mechanics, is to consider $f(x) = -\dfrac{dV(x)}{dx}$.

  where $V(x)$ is a potential energy function.

- Imagine a particle sliding down the potential $V(x)$ which has a layer of goo on the walls.



i.e. the particle is heavily damped, its inertia is negligible ($m\ddot{x} \ll b\dot{x}$). where b is the resistive force due to the goo).

Using the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}$$

For the first order system

$$\frac{dx}{dt} = f(x) = -\frac{dV}{dx}$$

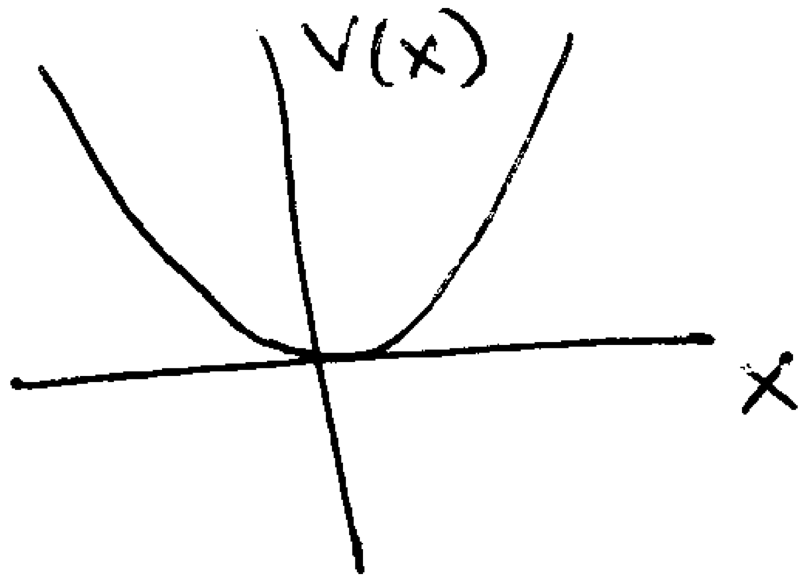
$$\text{then } \frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0$$

- Hence $V(t)$ decreases along trajectories and the particle always moves to lower potential.
- If particle is at equilibrium pt, $\frac{dV}{dx} = 0$, then $V \equiv c$
- Note the local minima (maxima) of $V(x)$ are stable (unstable) fixed points.

Ex. Graph potential & find equilibrium pts.

for $\dot{x} = -x$.

- $-\frac{dV}{dx} = -x \Rightarrow V = \frac{1}{2}x^2 + C$ (usually let $C \neq 0$)

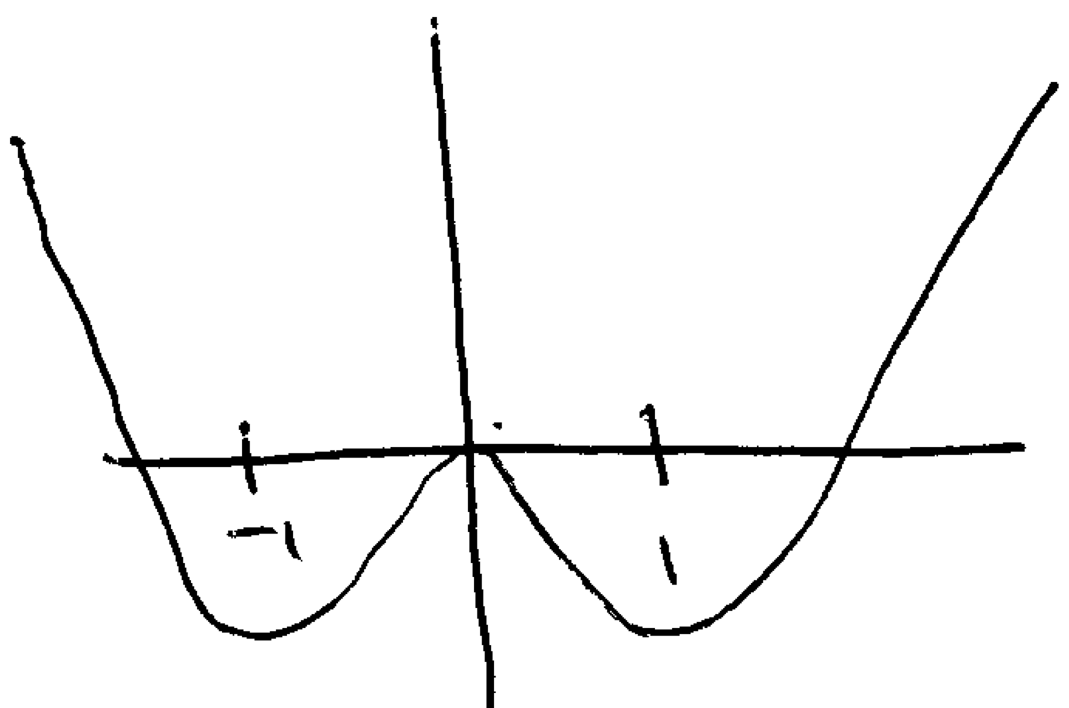


the only equilibrium occurs at $x=0$; it's a minimum of $V(x)$
$\Rightarrow x=0$ is a stable fixed pt.

Ex. $\dot{x} = x - x^3$

- $-\frac{dV}{dx} = x - x^3$

$\Rightarrow V = -\frac{1}{2}x^2 + \frac{x^4}{4} + C$; let $C=0$



double-well potential

- local minima at $x = \pm 1$ correspond to stable equilibria
- local max at $x=0$ corresponds to unstable equilibrium.