4. Linear and Nonlinear Systems in 2D

In higher dimensions, trajectories have more room to manoeuvre, and hence a wider range of behaviour is possible.

4.1 Linear systems: definitions and examples

A 2-dimensional linear system has the form

$$\begin{aligned}\dot{x} &= ax + by \\ \dot{y} &= cx + dy\end{aligned}$$

where a, b, c, d are parameters. Equivalently, in vector notation

$$\dot{\mathbf{x}} = \underline{\mathbf{A}}\mathbf{x} \tag{1}$$

where

$$\underline{\mathbf{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

The *Linear* property means that if $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions, then so is $c_1\mathbf{x_1} + c_2\mathbf{x_2}$ for any c_1 and c_2.

The solutions of $\dot{x} = \underline{\mathbf{A}}\mathbf{x}$ can be visualized as trajectories moving on the (x, y) plane, or *phase plane*.

Example 4.1.1 $m\ddot{x} + kx = 0$ i.e. the simple harmonic oscillator

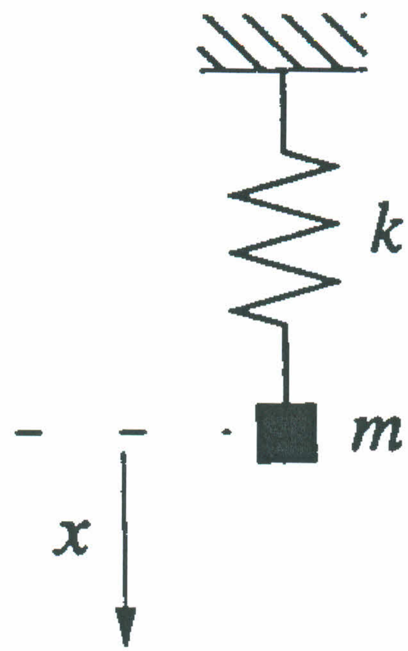


Fig. 4.1.1

The state of the system is characterized by x and $v = \dot{x}$

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{k}{m}x \end{aligned}$$

i.e. for each (x, v) we obtain a vector $(\dot{x}, \dot{v}) \Rightarrow$ **vector field** on the phase plane.

As for a 1-dimensional system, we imagine a fluid flowing steadily on the phase plane with a local velocity given by $(\dot{x}, \dot{v}) = (v, -\omega^2 x)$.

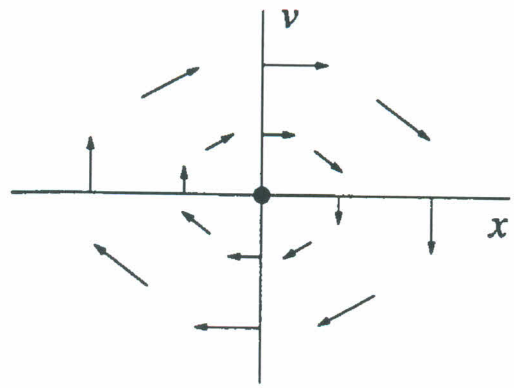


Fig. 4.1.2

- Trajectory is found by placing an imaginary particle or *phase point* at (x_0, v_0) and watching how it moves.

- $(x, v) = (0, 0)$ is a fixed point:
  static equilibrium!

- Trajectories form closed orbits around $(0, 0)$:
  oscillations!

The *phase portrait looks like...*

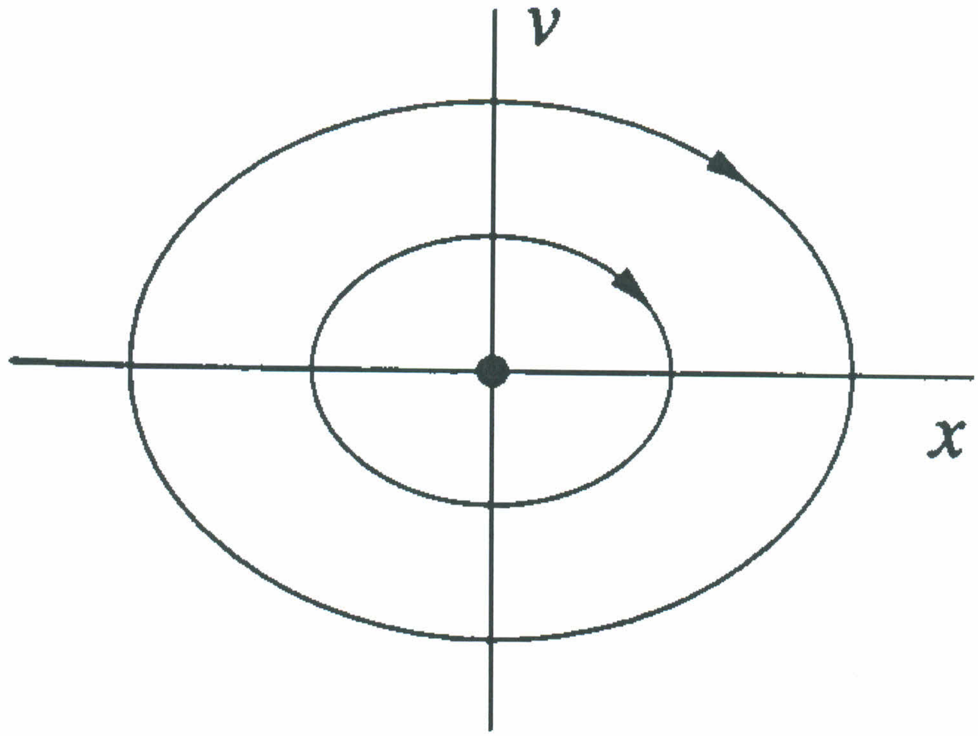


Fig. 4.1.3

- NB $\omega^2 x^2 + v^2$ is *constant* on each ellipse. This is simply the *energy*

<u>Example 4.1.2</u>

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The phase portraits for these *uncoupled* equations are...

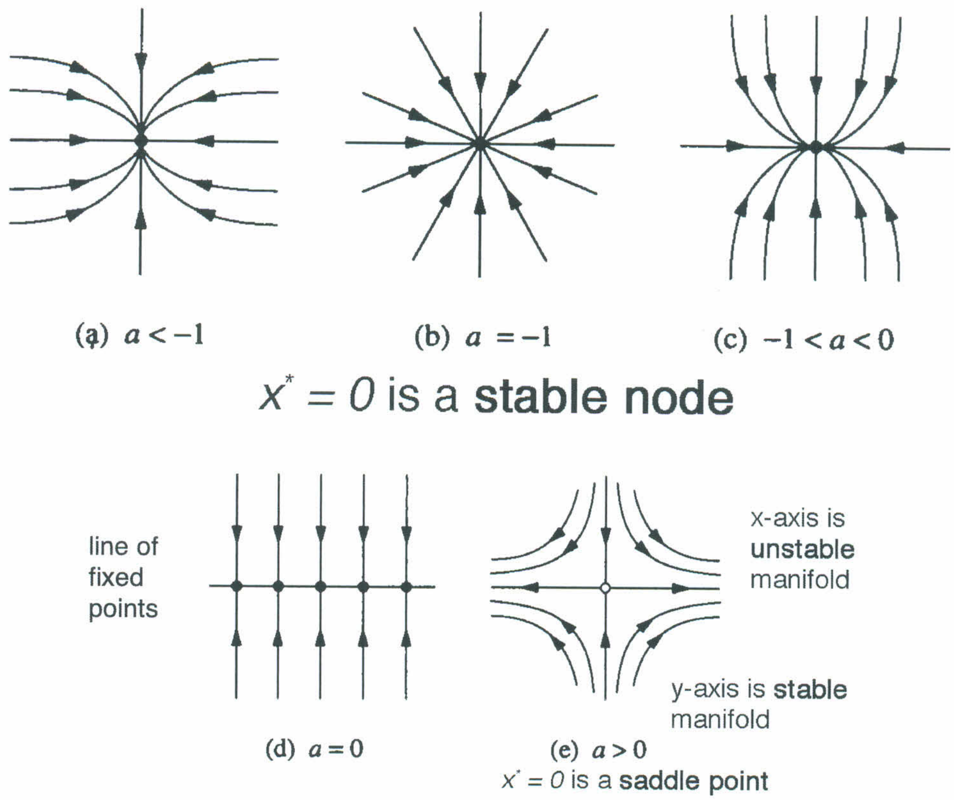


Fig. 4.1.4

Solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 e^{at} \\ y_0 e^{-t} \end{pmatrix}$$

Some terminology...

- $x^* = 0$ is an *attracting* fixed point in Figs (a) - (c) since $x(t) \to x^*$ as $t \to \infty$.

- $x^* = 0$ is called **Lyapunov Stable** in Figs (a) - (d) since all trajectories that start sufficiently close to x^* remain close to it for all time.

- Fig. (d) shows that a *fixed point* can be *Lyapunov stable but not attracting* $\Rightarrow$ it is *neutrally stable*. It is also possible for a fixed point to be attracting but not Lyapunov stable!

- If a fixed point is *both* Lyapunov stable *and* attracting, we'll call it *stable*, or sometimes *asymptotically stable*

- $\mathbf{x}^*$ is unstable in Fig. (e) because it is *neither* attracting nor Lyapunov stable

4.2 Classification of Linear Systems

Consider a general 2 ×2 matrix $\underline{\mathbf{A}}$ such that $\dot{\mathbf{x}} = \underline{\mathbf{A}}\mathbf{x}$

To solve: try

$$
\begin{aligned}
\mathbf{x}(t) &= e^{\lambda t}\mathbf{v} \quad (\mathbf{v} \text{ is a constant vector}) \\
\Rightarrow \lambda e^{\lambda t}\mathbf{v} &= e^{\lambda t}\underline{\mathbf{A}}\mathbf{v} \\
\Rightarrow \underline{\mathbf{A}}\mathbf{v} &= \lambda\mathbf{v}
\end{aligned}
$$

Hence if we obtain the eigenvectors $\mathbf{v}$ and eigenvalues λ, we will have two independent solutions $\mathbf{x}(t)$. Recall that $\underline{\mathbf{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has eigenvalues λ_1 and λ_2, where

$$
\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2} \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}
$$

$$
\begin{aligned}
\text{with} \quad \tau &= \text{trace}(\underline{\mathbf{A}}) &= a + d \\
\Delta &= \det(\underline{\mathbf{A}}) &= ad - bc
\end{aligned}
$$

$$\vec{x}(t) = e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\dot{\vec{x}} = A\vec{x}$$

$$\lambda e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = A e^{\lambda t}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

or

$$\lambda\vec{v} = A\vec{v}$$

- so desired st. line soln exists if v is an e.vector for A with eigenvalue λ.

Let's recall how to find e.values/e.vectors.

want a nontrivial soln to $Ax = \lambda x$

$$(A - \lambda I)x = 0$$

$$\Rightarrow \det(A - \lambda I) = 0$$ for nontrivial soln (characteristic eqn)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} = 0$$

$$(a-\lambda)(d-\lambda) - bc = 0$$

$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

or $\lambda^2 - \tau\lambda + \Delta = 0$ where $\tau = \text{tr} A = a+d$, $\Delta = \det A = ad - bc$.

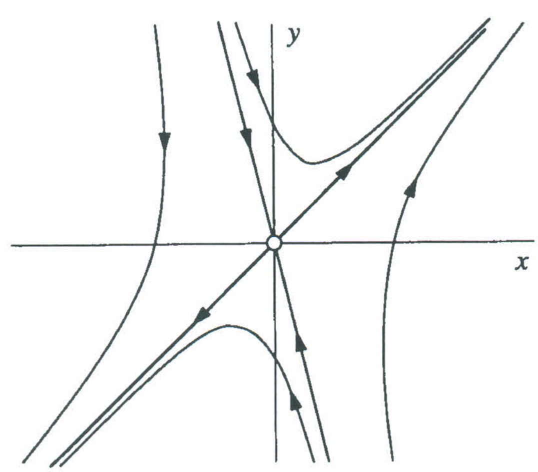
then $$\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

- Useful check when calculating eigenvalues: $\lambda_1 + \lambda_2 = \tau$ and $\lambda_1\lambda_2 = \Delta$

<u>Example 4.2.1</u> $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

- $\Rightarrow \lambda_1 = 2$ with $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \lambda_1 > 0$
  hence solution *grows*

- $\Rightarrow \lambda_2 = -3$ with $\mathbf{v_2} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ $\quad \lambda_2 < 0$
  hence solution *decays*



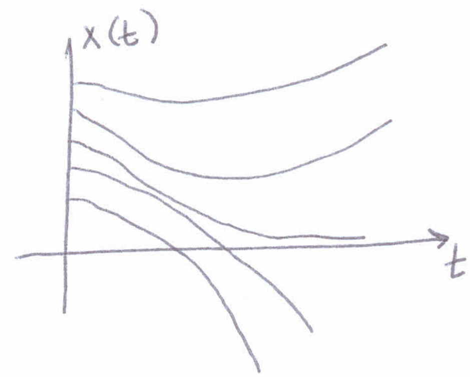


Fig. 4.2.1

Notice $\lambda_1 + \lambda_2 = \frac{\tau + \sqrt{\quad}}{2} + \frac{\tau - \sqrt{\quad}}{2}$

$= \tau$

$\lambda_1 \lambda_2 = \left(\frac{\tau + \sqrt{\quad}}{2}\right)\left(\frac{\tau - \sqrt{\quad}}{2}\right)$

$= \frac{\tau^2 - (\tau^2 - 4\Delta)}{4} = \Delta.$

so can use this as a check

Typical situation ~~is~~ for $\lambda_1 \neq \lambda_2$. In this case, the corr. evectors $\vec{v}_1, \vec{v}_2$ are lin ind & span the entire plane.

So <u>any</u> i.c. $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$ (can be written as linear comb.)

$\Rightarrow$ general soln.

$x(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$

<u>why?</u> 1) it is a linear comb of solns to $\dot{x} = Ax$ & hence is a soln.

2) It satisfies the i.c. $\vec{x}(0) = \vec{x}_0$

so by existence & uniqueness thm, this the only soln.

8c.

$$\dot{x} = x + y$$
$$\dot{y} = 4x - 2y$$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow \det\begin{pmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{pmatrix} = 0$$

$$\lambda^2 + \lambda - 2 - 4 = 0$$
$$\lambda^2 + \lambda - 6 = 0$$
$$(\lambda + 3)(\lambda - 2) = 0$$
$$\lambda = -3, 2.$$

for $\lambda = -3$

$$(A - \lambda I)\vec{v} = 0$$
$$\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$
$$4v_1 + v_2 = 0$$
$$v_2 = -4v_1$$

nontrivial soln $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$

for $\lambda = 2$.

$$\begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$
$$-v_1 + v_2 = 0 \Rightarrow v_1 = v_2$$

nontrivial soln $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow x(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix} e^{-3t}$$

If given initial data $(x_0, y_0) = (2, -3)$

$$\begin{pmatrix} 2 \\ -3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \quad \text{or} \quad \begin{aligned} 2 &= c_1 + c_2 \\ -3 &= c_1 - 4c_2 \\ 5 &= 5c_2 \Rightarrow c_2 = 1 \\ & c_1 = 1 \end{aligned}$$

$$x(t) = \begin{pmatrix} e^{2t} + e^{-3t} \\ e^{2t} - 4e^{-3t} \end{pmatrix}$$

- *straight line trajectories* in Fig. 4.2.1 are the eigenvectors $\mathbf{v_1}$ and $\mathbf{v_2}$

<u>Example 4.2.2</u> Consider $\lambda_2 < \lambda_1 < 0$

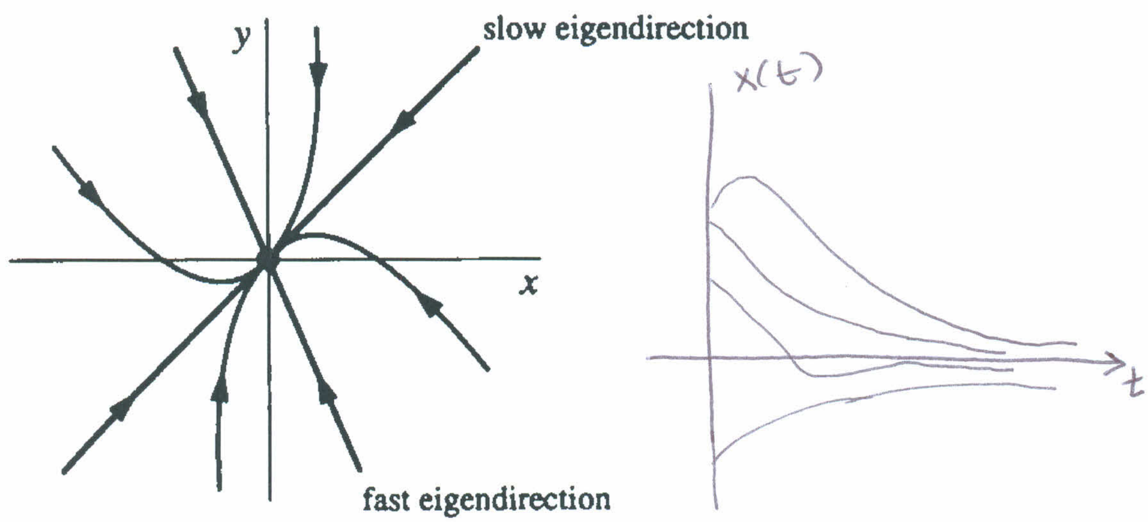


Fig. 4.2.2

- Both solutions decay exponentially!

Ex. $\frac{dx}{dt} = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{x}$

$\det(A - \lambda I) = \det\begin{pmatrix} 5-\lambda & -1 \\ 3 & 1-\lambda \end{pmatrix} = 0$

$(5-\lambda)(1-\lambda) + 3 = 0$

$\lambda^2 - \text{tr}A\,\lambda + \det A = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$(\lambda - 4)(\lambda - 2) = 0$

$\lambda = 4, 2$

$\lambda = 2$ $\quad \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$x_2 = 3x_1$

$\lambda = 4$ $\quad \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x_1 = x_2$

$x(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} = e^{4t}\left[c_1 e^{-2t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$