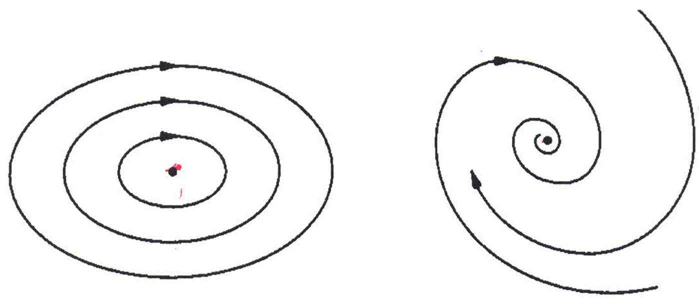
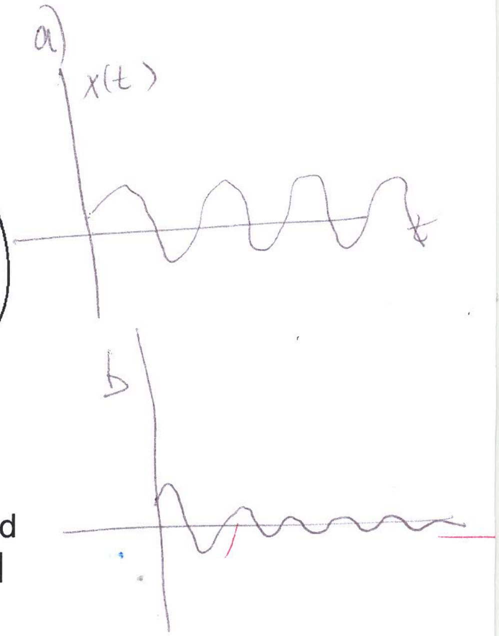
Lect Nov. 1st

Example 4.2.3 What happens if λ_1, λ_2 are complex?

Fixed point is either...





(a) center
[e.g. harmonic oscillator]
centre **neutrally** stable

(b) spiral
[e.g. lightly damped harmonic oscillator]

Fig. 4.2.3

- If λ_1, λ_2 are purely imaginary, all solutions are *periodic*

- If $\lambda_1 = \lambda_2$ we get a *star node* or a *degenerate node*

Last time considered phase plane for the case of 2 distinct real roots of either opposite sign, or the same sign.

Next, we examine when λ_1, λ_2 are complex

Q: what happens when λ are complex?

- get a center or a spiral
- centers are neutrally stable since nearby trajs are neither attracted nor repelled.

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right) = \alpha \pm i\omega \qquad \alpha = \tau/2$$

$$\omega = \frac{1}{2}\sqrt{4\Delta - \tau^2}$$

for $\mathbb{C}$ λ, $\tau^2 - 4\Delta < 0$

- since $\omega \neq 0$, 2 distinct λ's

$$x(t) = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2}$$

v_i's are eigenvectors for λ_i

where C's and $\vec{v}$'s are complex

$x(t)$ involves linear comb of $e^{(\alpha \pm i\omega)t}$

By Eulers: $e^{i\omega t} = \cos\omega t + i\sin\omega t$

$e^{\alpha t}\cos\omega t$, $e^{\alpha t}\sin\omega t$ represent exp decaying oscillations if $\alpha < 0$

exp. growing oscillations if $\alpha > 0$

So obtain stable ($\alpha < 0$), unstable ($\alpha > 0$) spirals

- If λ's pure imaginary, $\alpha = 0$
  all solns are periodic with $T = 2\pi/\omega$
  $\vec{x}(t) = c_1 \cos\omega t\, \vec{v}_1 + c_2 \sin\omega t\, \vec{v}_2$
  oscillations have fixed amplitude & fixed pt is a center.

Alternately, • Systems with eigenvalues $\lambda = \alpha \pm i\omega$ are typified by

$$\dot{\vec{x}} = \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} \vec{x}$$

or

$$(1) \quad \begin{aligned} \dot{x}_1 &= \alpha x_1 + \omega x_2 \\ \dot{x}_2 &= -\omega x_1 + \alpha x_2 \end{aligned}$$

- introduce polar coords $\begin{aligned} x_1 &= r\cos\theta \\ x_2 &= r\sin\theta \end{aligned}$, $\begin{aligned} r^2 &= x_1^2 + x_2^2 \\ \tan\theta &= \frac{x_2}{x_1} \end{aligned}$ $(\bigstar)$

- then differentiating $\bigstar$ w.r.t. t obtain

$$(\bigstar\bigstar) \quad r\dot{r} = x_1\dot{x}_1 + x_2\dot{x}_2 \qquad \sec^2\theta\, \dot{\theta} = \frac{x_1\dot{x}_2 - x_2\dot{x}_1}{x_1^2}$$

- substituting (1) into $(\bigstar\bigstar)$

$$\dot{r} = \frac{x_1(\alpha x_1 + \omega x_2) + x_2(-\omega x_1 + \alpha x_2)}{r}$$

$$= \frac{\alpha x_1^2 + \alpha x_2^2}{r} = \frac{\alpha r^2}{r} = \alpha r$$

$$\Rightarrow r = ce^{\alpha t}$$

similarly, $\sec^2\theta = r^2/x_1^2$

$$\dot{\theta} = -\omega$$

$$\Rightarrow \theta = -\omega t + \theta_0$$

- parametric eqns in polar coords of traj:

$$r = ce^{\alpha t}$$

$$\theta = -\omega t + \theta_0$$

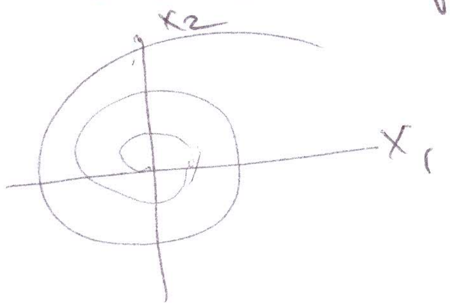
Since $\omega > 0$, θ decreases as t increases

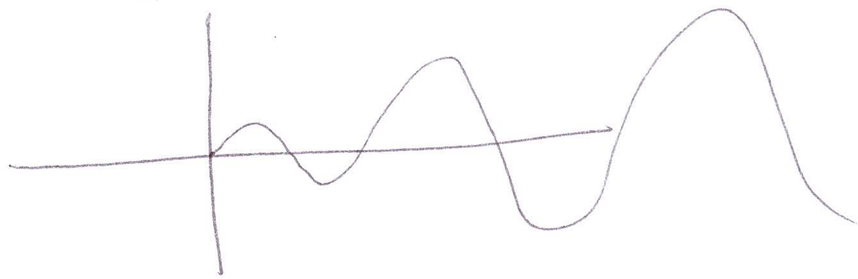
so motion clockwise.

As $t \to \infty$, $r \to 0$ if $\lambda < 0$

$r \to \infty$ $\lambda > 0$

$\Rightarrow$ trajs are spirals which approach or recede from origin depen λ.





---

pure imag λ: $\dot{r} = 0$, $\dot{\theta} = -\omega$

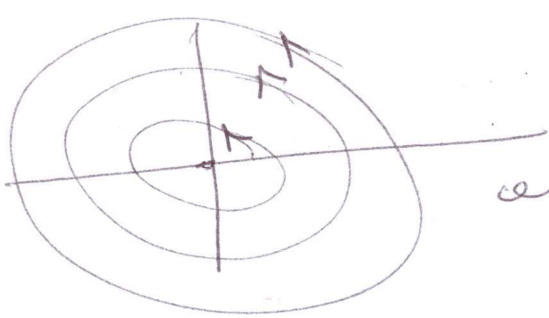
$\dot{\vec{x}} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \vec{x}$ $\quad r = C \quad \theta = -\omega t + C.$

$\lambda = \pm i\omega$

$\Rightarrow$ trajs are circles w center at origin, traversed clockwise if $\omega > 0$

counterclockwise $\omega < 0$.

$T = 2\pi/\omega$



center.

10d'

pure imag λ: $\dot{\vec{x}} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix} \vec{x}$, $\lambda = \mp i\omega$

$$\dot{x}_1 = \omega x_2$$
$$\dot{x}_2 = -\omega x_1$$

$$r\dot{r} = x_1 \dot{x}_1 + x_2 \dot{x}_2$$
$$= \omega x_2 x_1 + x_2(-\omega x_1) = 0$$

$$\Rightarrow r = C$$

$$\sec^2\theta \, \dot{\theta} = \frac{x_1 \dot{x}_2 - x_2 \dot{x}_1}{x_1^2}$$

$$\frac{r^2}{x_1^2} \dot{\theta} = \frac{x_1(-\omega x_1) - x_2(\omega x_2)}{x_1^2}$$

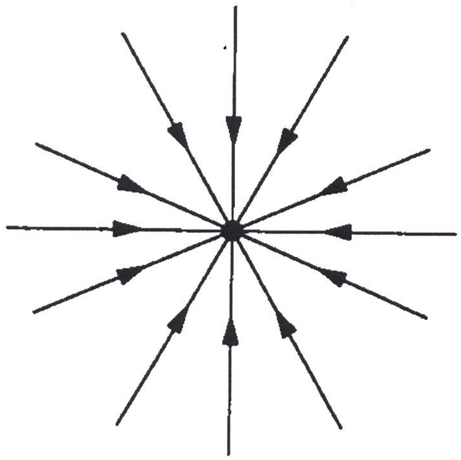
$$\dot{\theta} = \frac{-\omega(r^2)}{x_1^2} \cdot \frac{x_1^2}{r^2}$$

$$\dot{\theta} = -\omega$$

$$\Rightarrow \theta = -\omega t + \theta_0$$

- In general, when λ pure imaginary trajectories are ellipses centered at origin

$\lambda < 0$



Classification of Fixed Points

$\lambda_{1,2} = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta})$, where
$\Delta = \lambda_1 \lambda_2$ and $\tau = \lambda_1 + \lambda_2$

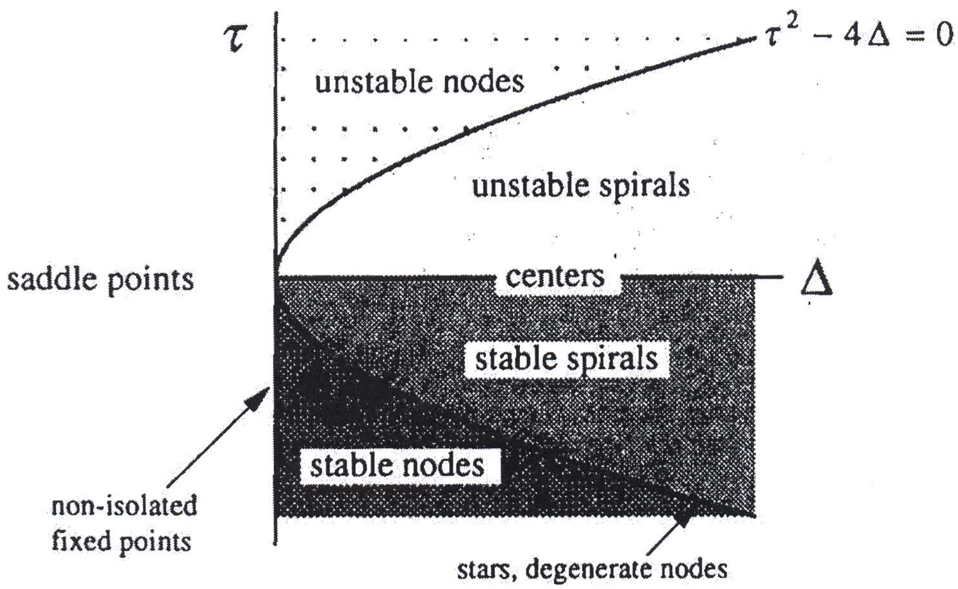


Fig. 4.2.4

Case of Equal Eigenvalues

Suppose $\lambda_1 = \lambda_2 = \lambda$.

- two subcases depending on whether the repeated λ has 2 independent e.vectors or only 1

- (a) Two independent eigenvectors $\vec{v}_1, \vec{v}_2$:

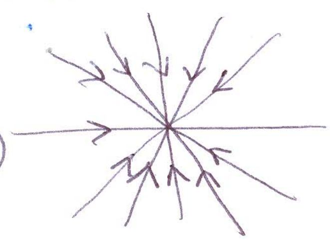
  then $\vec{x} = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}\vec{v}_2$

  the ratio $\dfrac{x_2}{x_1} = \dfrac{c_1 e^{\lambda t} v_{11} + c_2 e^{\lambda t} v_{21}}{c_1 e^{\lambda t} v_{12} + c_2 e^{\lambda t} v_{22}} = \dfrac{c_1 v_{11} + c_2 v_{21}}{c_1 v_{12} + c_2 v_{22}} = \alpha$

  ind of t

  $\Rightarrow$ every trajectory lies on a straight line thru the origin $x_2 = \alpha x_1$.

  $\lambda < 0$



- (b) one independent e.vector $\vec{v}$:

  $\vec{x} = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(\vec{v}t + \vec{w})$

  (w is the generalized e.vector assoc. w λ)

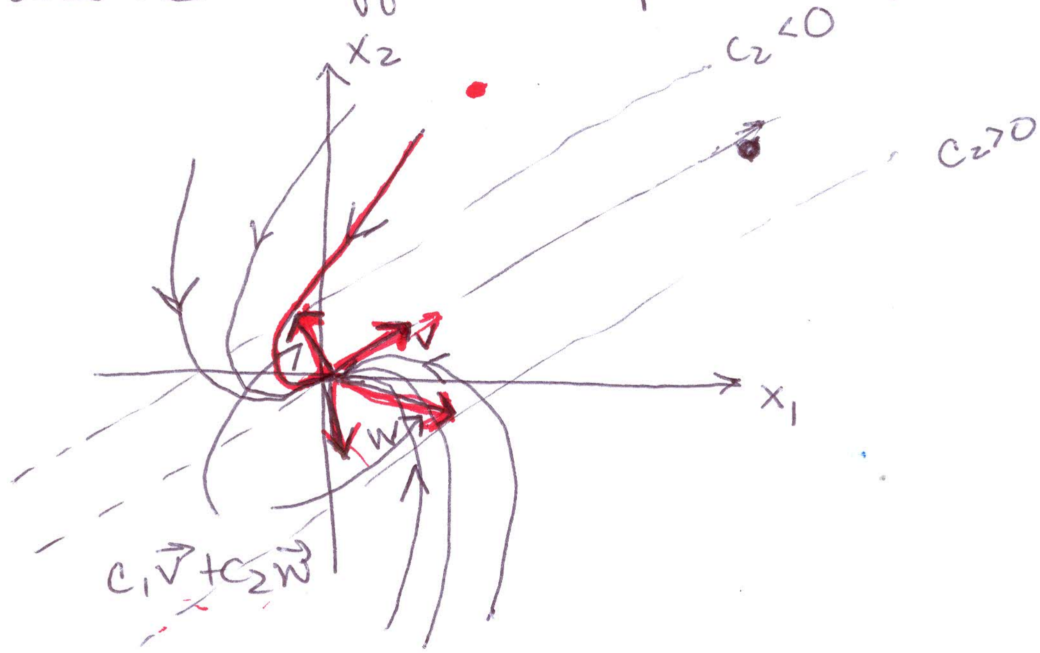
For large t, the dominant term is $c_2 e^{\lambda t}\vec{v}t$

Thus as $t \to \infty$, every trajectory approaches the origin tangent to the line thru the e.vector.

- orientation of the trajs depends on relative positions of $\vec{v}$ and $\vec{w}$.

$$\vec{x}(t) = \left[(c_1\vec{v} + c_2\vec{w}) + \underline{c_2\vec{v}\,t}\right]e^{\lambda t} = \vec{y}\,e^{\lambda t}$$

the vector $\vec{y}$ determines the direction of $\vec{x}$ whereas $e^{\lambda t}$ affects only the magnitude.



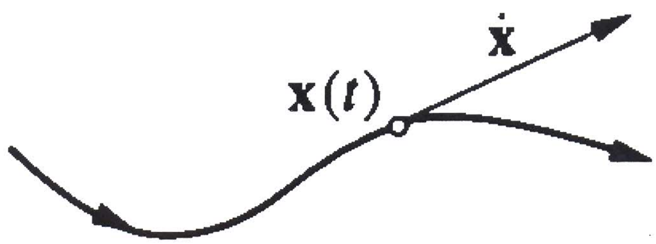
4.3 Phase Portraits

Recall $\dot{\mathrm{x}} = f(\mathrm{x})$, i.e.

$$\dot{x_1} = f_1(x_1, x_2)$$
$$\dot{x_2} = f_2(x_1, x_2)$$

• can now be nonlinear

where $\mathrm{x} = (x_1, x_2)$ and $f(\mathrm{x}) = (f_1(\mathrm{x}), f_2(\mathrm{x}))$ (not necessarily linear now). The trajectories $\mathrm{x}(t)$ wind their way through the phase plane.



$\dot{x}$ is the velocity vector at pt $\dot{x}(t)$.

The entire phase plane is filled with trajectories!

4.4 Example of a phase portrait

- Shows a sample of the *qualitatively different* trajectories

- For nonlinear systems typically no hope of finding trajectories analytically

• Want to find the phase portrait directly from the properties of $f(\vec{x})$.

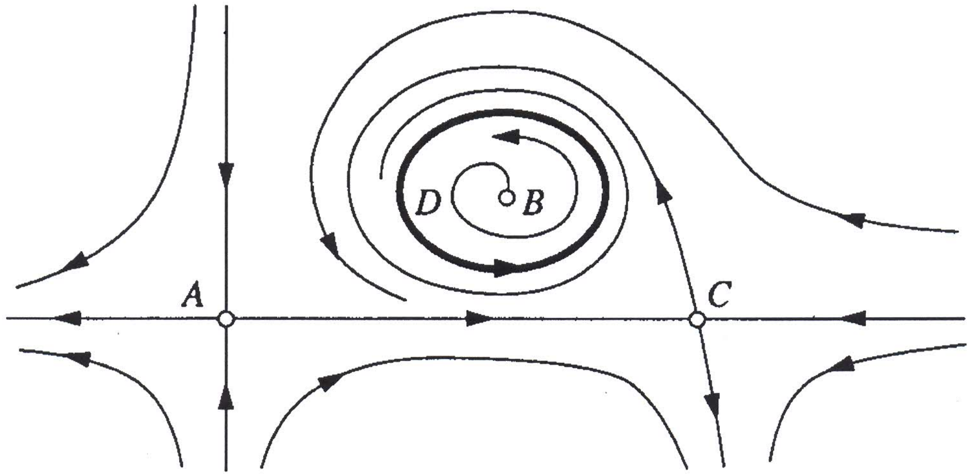


Fig. 4.4.1

- Fixed points A, B and C satisfy $f(\mathbf{x}^*) = 0$ and correspond to steady states or equilibria

- Closed orbit D corresponds to periodic solutions, i.e. $\mathbf{x}(t+T) = \mathbf{x}(t)$ for all t for some $T > 0$

- The existence and uniqueness theorem given for 1-dimensional systems can be generalized to 2-dimensional systems ... fortunately $\Rightarrow$ *different trajectories never intersect!*