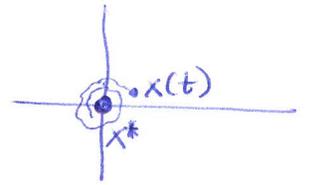


Linear Stability Analysis - 1D

rate of decay to a stable fixed pt  $x^*$  is obtained by linearizing abt  $x^*$ :



Let  $\eta(t) = x(t) - x^*$  be a small pert of  $x^*$

$$\dot{\eta} = \dot{x}$$

want to find the behavior  $\eta(t)$

$$\begin{aligned} \dot{\eta} &= \dot{x} = f(x) = f(x^* + \eta) \\ &= f(x^*) + \eta f'(x^*) + O(\eta^2) \\ &= \eta f'(x^*) + O(\eta^2). \end{aligned}$$

$$\Rightarrow \dot{\eta} \approx \eta f'(x^*)$$

linear eqn in  $\eta$ ,  
its stability:

sign of  $f'(x^*)$  determines

$f'(x^*) < 0$  stable

$f'(x^*) > 0$  unstable

$f'(x^*) = 0$  use different technique

# Conservative systems

$$m\ddot{z} = -\frac{dV}{dz}$$

$$m\dot{z}\ddot{z} = -\dot{z}\frac{dV}{dz}$$

note  $\frac{d}{dt} \frac{m(\dot{z})^2}{2} = \frac{m}{2} \cdot 2\dot{z}\ddot{z}$

$$\frac{d}{dt} V(z) = \frac{dV}{dz} \frac{dz}{dt} = \frac{dV}{dz} \dot{z}$$

$$\frac{d}{dt} \left( \frac{m\dot{z}^2}{2} + V(z) \right) = 0$$

$$\frac{m\dot{z}^2}{2} + V(z) = C = E(z)$$

Taylor's thm for function of 2 variables:

- assume all partial derivatives of order  $n$  are continuous in a closed region, all  $(n+1)$  partials exist in the open region, then

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f(x_0, y_0) + \frac{1}{2!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f(x_0, y_0) + \dots + \frac{1}{n!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n f(x_0, y_0) + R_n$$

$$R_n = \frac{1}{(n+1)!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(x_0 + \theta h, y_0 + \theta k)$$

where  $\left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right) f = h f_x + k f_y$   
 $\left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^2 f = h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}$

$\left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n$  expanded using binomial theorem.

alternate notation:  $h = \Delta x = x - x_0$

$k = \Delta y = y - y_0$

$h = x - 1, k = y - \frac{\pi}{2}$

Ex. expand  $f(x, y) = \sin xy$  in powers of  $x-1$  and  $y - \frac{\pi}{2}$  to second degree

$$f_x = y \cos xy$$

$$f_y = x \cos xy$$

$$f_{xx} = -y^2 \sin xy$$

$$f_{xy} = -xy \sin xy$$

$$f_{yy} = -x^2 \sin xy$$

$$f(x, y) \approx f(1, \pi/2) + (x-1) f_x(1, \pi/2) + (y - \pi/2) f_y(1, \pi/2)$$

$$+ \frac{1}{2} \left( (x-1)^2 f_{xx}(1, \pi/2) + 2(x-1)(y - \pi/2) f_{xy}(1, \pi/2) + (y - \pi/2)^2 f_{yy}(1, \pi/2) \right)$$

$$= 1 + 0 + 0 + \frac{1}{2} \left( \left( \frac{\pi}{2} \right)^2 (x-1)^2 - \frac{\pi}{2} (x-1)(y - \pi/2) - (y - \pi/2)^2 \right)$$

## 4.5 Fixed points and Linearization

This is the same idea as for 1-dimensional systems

$$\begin{aligned}\dot{x} &= f(x, y) \\ \dot{y} &= g(x, y)\end{aligned}$$

Suppose  $(x^*, y^*)$  is a fixed point. Expand around  $(x^*, y^*)$  using  $u = x - x^*$  and  $v = y - y^*$ . *u, v small*

$$\begin{aligned}\dot{u} = \dot{x} &= f(x^* + u, y^* + v) \\ &= f(x^*, y^*) + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + O(u^2, v^2, uv) \\ &\simeq u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y}\end{aligned}$$

Similarly

$$\dot{v} \simeq u \frac{\partial g}{\partial x} + v \frac{\partial g}{\partial y}$$

Hence a *small* disturbance around  $(x^*, y^*)$  evolves as

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

where the matrix is known as the *Jacobian matrix*  $\underline{\mathbf{A}}$  at  $(x^*, y^*)$ , and is the multivariable equivalent of  $f'(x^*)$  for 1-D systems.

### Example 4.5.1

$$\begin{aligned}\dot{x} &= -x + x^3 \\ \dot{y} &= -2y\end{aligned}$$

Fixed points occur where  $\dot{x} = 0$  and  $\dot{y} = 0$  simultaneously. Hence  $x = 0$  or  $x = \pm 1$  and  $y = 0 \Rightarrow 3$  fixed points  $(0, 0)$ ,  $(1, 0)$  and  $(-1, 0)$

Jacobian matrix  $\underline{\mathbf{A}}$

$$\underline{\mathbf{A}} = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{pmatrix}$$

At  $(0, 0)$   $\underline{\mathbf{A}} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow$  stable node

At  $(\pm 1, 0)$   $\underline{\mathbf{A}} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow$  both are saddle points.

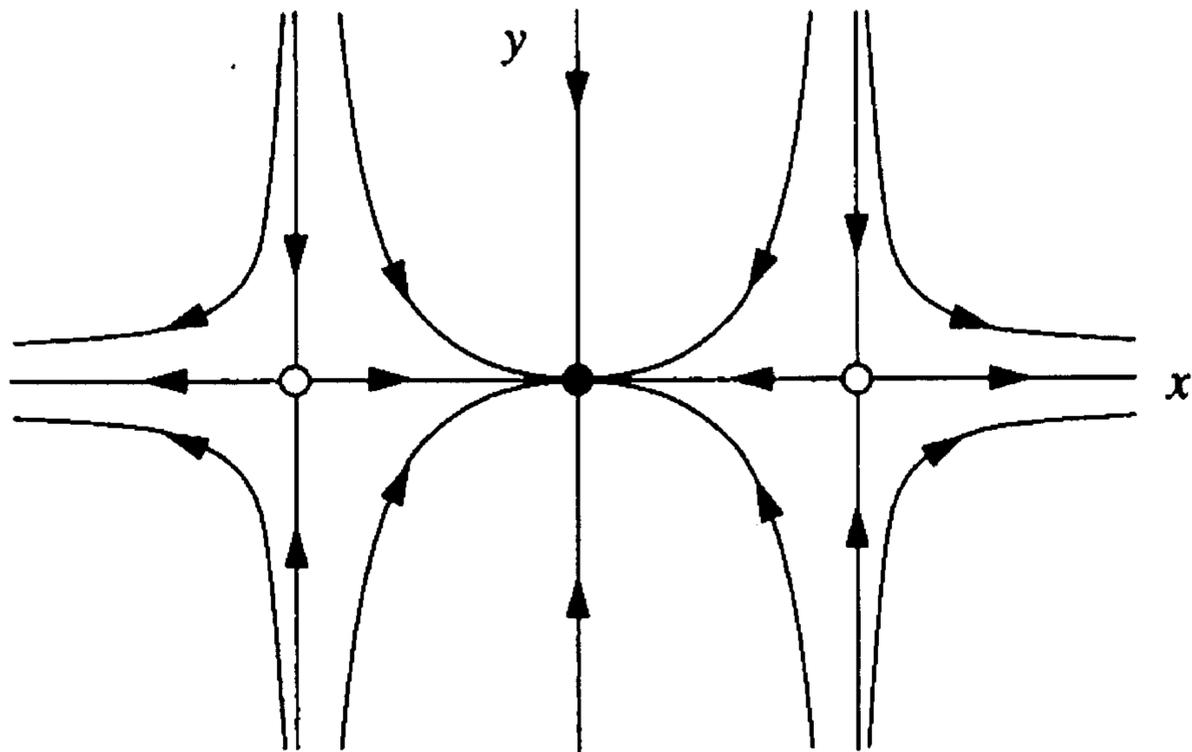


Fig. 4.5.1

In general, we must obtain fixed points by solving  $\dot{x} = 0$  and  $\dot{y} = 0$  simultaneously.

$$\begin{aligned} \text{e.g. } \dot{x} &= x(3 - x - 2y) \\ \dot{y} &= y(2 - x - y) \end{aligned}$$

yields fixed points  $(0, 0)$ ,  $(0, 2)$ ,  $(3, 0)$  and  $(1, 1)$

In general,  $\underline{\mathbf{A}}$  will not be diagonal at  $(x^*, y^*)$ . Hence we must diagonalize  $\underline{\mathbf{A}}$ , i.e. *find eigenvalues  $\lambda_1$  and  $\lambda_2$  and eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  of  $\underline{\mathbf{A}}$*

Basically, we are doing the same here as before for 2D *linear* systems, since we are treating the *nonlinear* system as *linear* near  $(x^*, y^*)$ . Knowledge of  $\lambda_1$  and  $\lambda_2$ , and  $v_1$  and  $v_2$ , enables us to sketch the phase portrait near  $(x^*, y^*)$ .

The fixed points can be *classified* according to their *stability* as follows:

- If  $\text{Re}(\lambda_1) > 0$  and  $\text{Re}(\lambda_2) > 0$   
 $\Rightarrow$  repeller (unstable node)
- If  $\text{Re}(\lambda_1) < 0$  and  $\text{Re}(\lambda_2) < 0$   
 $\Rightarrow$  attractor (stable node)
- If  $\text{Re}(\lambda_1) > 0$  but  $\text{Re}(\lambda_2) < 0$  (or vice versa)  $\Rightarrow$  saddle
- If  $\lambda_1$  and  $\lambda_2$  are both imaginary  $\Rightarrow$  centre

## 4.6 Example: Rabbits vs Sheep

An example of the *Lotka-Volterra* model of competition between two species (e.g. rabbits and sheep) grazing the same food supply (grass).

- Each species grows to its carrying capacity in the absence of the other - *logistic growth* (rabbits faster...!)
- When species encounter each other, the larger (sheep) has an advantage.
- Conflicts occur at a rate proportional to the size of each population. Conflicts reduce the growth rate of each species (but more for rabbits).

A model encapsulating these properties could be (see above!)

$$\begin{aligned}\dot{x} &= x(3 - x - 2y) \\ \dot{y} &= y(2 - x - y)\end{aligned}$$

Fixed points at

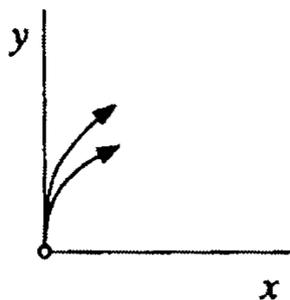
$$(0, 0) \quad \text{where} \quad \underline{\mathbf{A}} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda = 3, 2$$

$$(0, 2) \quad \text{where} \quad \underline{\mathbf{A}} = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix} \Rightarrow \lambda = -1, -2$$

$$(1, 1) \quad \text{where} \quad \underline{\mathbf{A}} = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix} \Rightarrow \lambda = -1 \pm \sqrt{2}$$

$$(3, 0) \quad \text{where} \quad \underline{\mathbf{A}} = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix} \Rightarrow \lambda = -3, -1$$

$(0, 0)$ :  $\lambda = 3, 2 \Rightarrow$  **unstable node** (repeller)



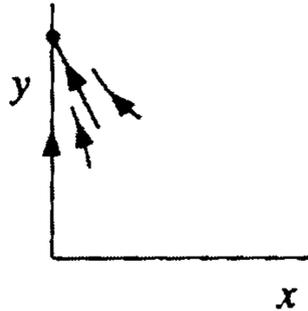
$$\lambda = 2 \Rightarrow \mathbf{v} = (0, 1) \quad \text{“slow eigendirection”}$$

$$\lambda = 3 \Rightarrow \mathbf{v} = (1, 0) \quad \text{“fast eigendirection”}$$

General rule...

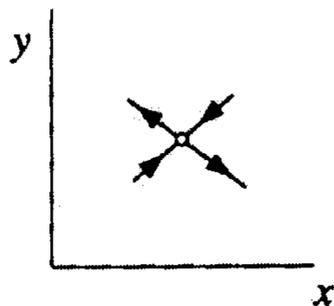
**Trajectories are tangential to the slow eigendirection (i.e. smallest  $|\lambda|$ ) at a node**

$(0, 2)$ :  $\lambda = -1, -2 \Rightarrow$  **stable node** (attractor)

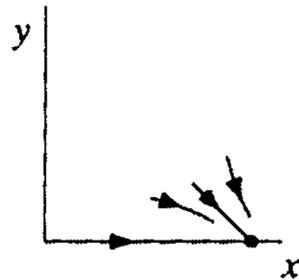


- Once again... Trajectories are tangential to the slow eigendirection at a node
- Here  $\lambda = -1 \Rightarrow \mathbf{v} = (1, -2)$  is the slow eigendirection.

$(1, 1)$ :  $\lambda = -1 \pm \sqrt{2} \Rightarrow$  **saddle point**



$(3, 0)$ :  $\lambda = -3, -1 \Rightarrow$  **stable node** (attractor)



Putting these together, the phase portrait becomes....

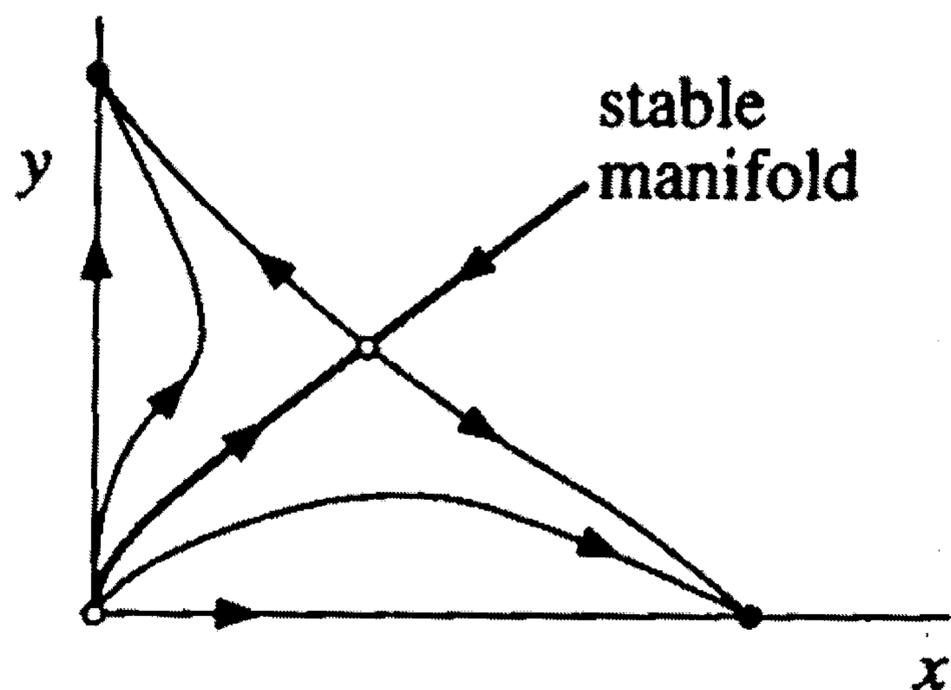


Fig. 4.6.1

NB: You don't really need to calculate the eigenvectors to get the right shape!

## Biological interpretation...

- In general, one species eventually drives the other to extinction; which species eventually dominates depends on initial populations  $\mathbf{x}_0 = (x_0, y_0)$
- **Basin of attraction** of an attracting fixed point  $x^*$  defined as the set of initial conditions  $\mathbf{x}_0$  such that  $\mathbf{x} \rightarrow x^*$  as  $t \rightarrow \infty$ .
- In this case, **basin boundary** is the stable manifold of the saddle node at  $(1, 1)$

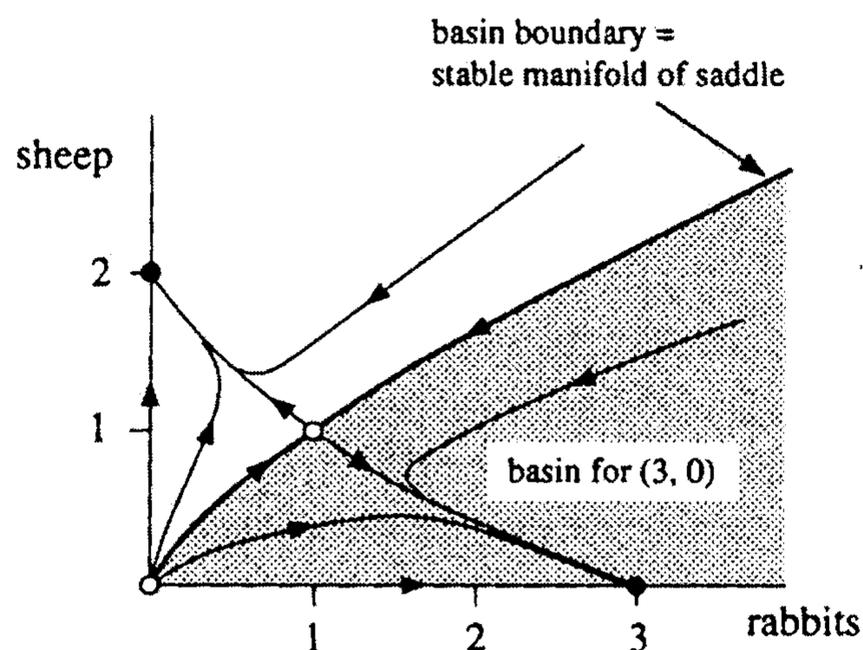


Fig. 4.6.2

## 4.7 Conservative Systems

Consider  $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ . A *conserved quantity* of this system is a real-valued continuous function  $E(\mathbf{x})$  that is *constant* on trajectories i.e.  $dE/dt = 0$ .

Example 4.7.1  $m\ddot{z} = -dV(z)/dz = F(z)$

Take  $x = z$  and  $y = \dot{z} \Rightarrow$

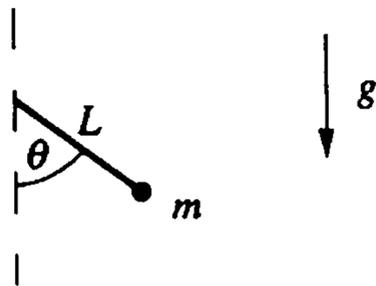
$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \frac{1}{m}F(x)\end{aligned}$$

$E(z) = \frac{1}{2}m\dot{z}^2 + V(z)$  is the total energy, which is *constant*

$$\begin{aligned}\Rightarrow E(\mathbf{x}) &\equiv \frac{1}{2}my^2 + V(x) \\ \frac{dE(\mathbf{x})}{dt} &= 0\end{aligned}$$

since total energy is constant.

Example 6.5.2  $\ddot{\theta} + \sin \theta = 0$



e.g. undamped simple pendulum

$$\dot{\theta} = \nu$$

$$\dot{\nu} = -\sin \theta$$

Fixed points at  $(\theta^*, \nu^*) = (k\pi, 0)$

$(0, 0)$  :  $\underline{\mathbf{A}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \lambda = \pm i \Rightarrow$  **centre**  
(oscillations)

Energy  $E(\theta, \nu) = \frac{1}{2}\nu^2 - \cos \theta$  is conserved, since

$$\frac{dE}{dt} = \nu \dot{\nu} + \sin \theta \dot{\theta} = \nu[\ddot{\theta} + \sin \theta] = 0$$

$(\pi, 0)$  :  $\underline{\mathbf{A}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda = \pm 1 \Rightarrow$  **saddle**

Phase portrait becomes...

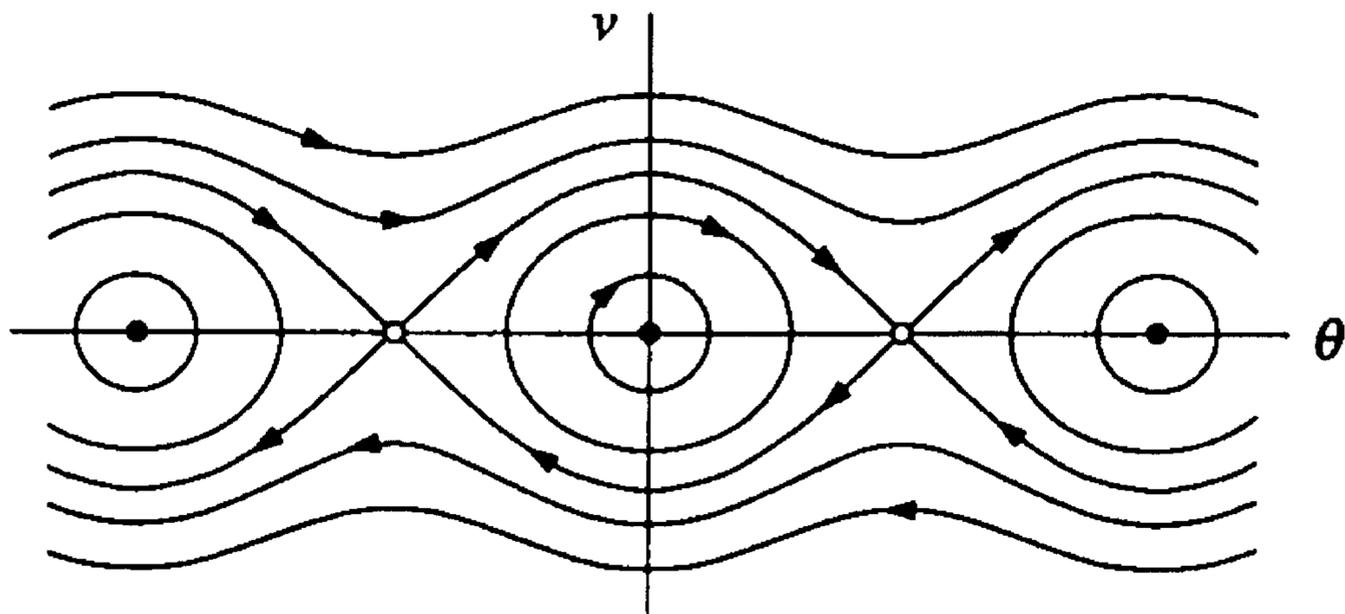


Fig. 4.7.1