A Brief Introduction to Control Theory and Applications

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Outline

1. Introduction
2. General Formulation
3. A Result of Controllability
4. An LQ Problems
5. Some Accessible Problems

1. Introduction

A Typical Example

An object, driven by an external force, is moving in $\mathbb{R}^3$.

$x(t)$ — position at time $t \geq 0$

$x(0)$ — initial position; $\dot{x}(0) = v_0$ — initial velocity

$u(t)$ — external force at time $t \geq 0$.

By Newton's second law, ($\mathbf{F} = m\mathbf{a}$)

$$\ddot{x}(t) = bu(t), \qquad t \geq 0, \tag{1.1}$$

with $b = \frac{1}{m} > 0$.

- For different choice $u(\cdot)$, the solution $x(\cdot)$ will be different
- $u(\cdot)$ is called a *control*, $x(\cdot)$ is called a (part of) *state*
- (1.1) is called a *control system*

Controllability Problems:

1. For any $x_0, x_1, v_0, v_1 \in \mathbb{R}^3$, find $T > 0$, $u : [0, T] \to \mathbb{R}^3$, s.t.

$$\begin{cases} x(0) = x_0, \quad \dot{x}(0) = v_0, \\ x(T) = x_1, \quad \dot{x}(T) = v_1. \end{cases} \tag{1.2}$$

(Controllability, without constraints)

2. What happens if the external force is not unlimited? e.g.,

$$\|u(t)\| \equiv \Big(\sum_{i=1}^{3} |u_i(t)|^2 \Big)^{\frac{1}{2}} \leq K. \tag{1.3}$$

(Controllability with control constraints)

3. What happens if $x(t)$ is restricted? e.g.,

$$x(t) \in G \subseteq \mathbb{R}^3, \qquad t \geq 0. \tag{1.4}$$

(Controllability with state constraints, motion planning)

Introduce cost functional:

$$J(u(\cdot))=\int_0^T\Big(\|x(t)-z(t)\|^2+\|u(t)\|^2\Big)dt+\|x(T)-\xi\|^2. \tag{1.5}$$

- first term — *running cost*
- second term — *terminal cost*
- $z(\cdot)$ — desired trajectory
- ξ — desired terminal state

Optimal Control Problems

1. Find $u(\cdot)$ such that $J(u(\cdot))$ is minimized
   (Optimal control problem without constraints)

2. What happens if there are constraints for state/control.
   (Optimal control problem with constraints)

Other Motivations

- Optimal Portfolio Selection:

Consider a market with n stocks and 1 bond.

$S_i(\cdot)$ — price process of Stock i $(1 \le i \le n)$

$S_0(\cdot)$ — price process of the bond

$N_i(\cdot)$ — share number of the i-th asset $(0 \le i \le n)$

Total wealth:

$$X(t) = \sum_{i=0}^{n} N_i(t) S_i(t) \equiv \sum_{i=0}^{n} u_i(t).$$

$u_i(t) \equiv N_i(t) S_i(t)$ — amount in the i-th asset

$u(\cdot) \equiv (u_1(\cdot), \cdots, u_n(\cdot))$ — a portfolio

From the dynamics of $S_i(\cdot)$ $(0 \leq i \leq n)$, one can derive a dynamics for $X(\cdot)$. Suppose the solution of this dynamics is given by

$$X(t) = \varphi(t; X_0, u(\cdot)), \quad t \geq 0, \qquad X(0) = X_0. \tag{1.6}$$

$X(t)$ — the value of portfolio at t $\quad u(\cdot)$ — the trading strategy

Goal: Find $u(\cdot)$ such that $X(T)$ is maximized

Since $X(\cdot)$ is a stochastic process, risk will be an important issue, we might want to maximize

$$J(u(\cdot)) = \mathbb{E}X(T) - \beta \operatorname{var}(X(T)),$$

for some $\beta > 0$ (Markovitz Mean-Variance Theory)

• Option Pricing: (Asset Pricing Problem)

A contract signed today (April 1):

> On June 1, the holder of the contract can come
>
> to buy one share of IBM stock at $130.

Let S be the price of IBM stock. On June 1.

(i) If $S > 130$, the holder will exercise the contract: come to buy and then sale immediately to make $S - 130$.

(ii) If $S < 130$, the holder will discard the contract.

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The contract holder has **right** only (no **obligation**).
Such a contract is called a *European call option*.

Sorry! There is no free lunch.

Question: What is the price of the contract?

At the mature time T, the profit of the option is

$$(S(T)-q)^+ = \begin{cases} S(T)-q, & S(T)>q, \\ 0, & S(T)<q. \end{cases}$$

$X(t)$ — fair price (value) of the option at t
$u(\cdot)$ — replicating strategy

$X(\cdot)$ follows some dynamic equation, whose solution is:

$$(1.6) \qquad X(t)=\varphi(t,X_0,u(\cdot)), \quad t\geq 0, \qquad X(0)=X_0.$$

Goal: Find $u(\cdot)$ such that $X(T)=(S(T)-q)^+$.

Note: $S(T)$ is not known at $t<T$. Need to "hit" a random target! (Black–Scholes Formula)

• Temperature Control: (population density)
$Y(t, x)$ — temperature at (t, x)
$u(t, x)$ — heat/cooling at (t, x) (harvesting)
Controlled heat transfer dynamics whose solution is given by:

$$(1.7) \qquad Y(t, x) = Y(t, x; u(\cdot)), \qquad (t, x) \in [0, \infty) \times G.$$

For given initial temperature distribution $Y_0(x)$, and terminal temperature distribution $Y_1(x)$, try to find $u(\cdot)$ such that

$$Y(0, x; u(\cdot)) = Y_0(x), \qquad Y(T; x, u(\cdot)) = Y_1(x).$$

This is impossible, in general. Need *approximate controllability*. Also, one can try to find $u(\cdot)$ to minimize

$$J(u(\cdot)) = \int_0^T \int_G \Big[|Y(t, x) - Z(x)|^2 + |u(t, x)|^2 \Big] dxdt$$

- Denoising Problem:

$Y(x)$ — Grey-level at $x \in G \subseteq \mathbb{R}^2$, a given picture.
Need to find $Z(x)$ piecewise smooth such that

$$\int_G |Y(x) - Z(x)|^2 dx + \int_G |\nabla Y(x)|$$

is minimized.

- Optical Flow Problem

- Shape from Shading

Many, many more...

2. General Formulation

Let $y(t) = \dot{x}(t)$. Then (1.1) can be written as

$$
(2.1) \qquad \begin{cases} \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t), \\ \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ v_0 \end{pmatrix}. \end{cases}
$$

More generally, we may consider

$$
(2.2) \qquad \begin{cases} \dot{X}(t) = AX(t) + Bu(t), \\ X(0) = X_0, \end{cases}
$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, $X(t)$ and $u(t)$ take values in $\mathbb{R}^n$ and $\mathbb{R}^m$. (2.2) is called a *linear control system*, denoted by $[A, B]$.

In general, we may consider controlled ODE:

$$\begin{cases} \dot{X}(s) = b(s, X(s), u(s)), \qquad s \in [0, T], \\ X(0) = X_0, \end{cases} \tag{2.3}$$

$X(\cdot)$ — state process, valued in $\mathbb{R}^n$

$u(\cdot)$ — control process, valued in U, a metric space

Let $\mathcal{U}[0, T] = \{ u : [0, T] \to U \bigm| u(\cdot) \text{ measurable} \}$.

Controllability Problem.

1. For any $X_0, X_1 \in \mathbb{R}^n$, find $T > 0$ and $u(\cdot) \in \mathcal{U}[0, T]$, such that

$$X(0) = X_0, \qquad X(T) = X_1. \tag{2.4}$$

2. For any $M_0, M_1 \subseteq \mathbb{R}^n$, find $T > 0$ and $u(\cdot) \in \mathcal{U}[0, T]$ such that

$$X(0) \in M_0, \qquad X(T) \in M_1. \tag{2.5}$$

Introduce cost functional

$$J(u(\cdot)) = \int_0^T g(s, X(s), u(s))ds + h(X(T)). \tag{2.6}$$

Problem (ODE). Find $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ such that

$$J(\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[0,T]} J(u(\cdot)). \tag{2.7}$$

Classical Calculus of Variations:

$$b(s, X, u) = u, \quad h(X) = 0 \tag{2.8}$$

Then Problem (ODE) is to minimize the following:

$$J(X(\cdot)) = \int_0^T g(s, X(s), \dot{X}(s))ds, \tag{2.9}$$

over a set of functions $X(\cdot)$. This is a classical calculus of variation problem.

3. A Result on Controllability

Consider system $[A, B]$:

$$\dot{X}(t) = AX(t) + Bu(t), \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$.

Theorem (Kalman). The following are equivalent:

(i) $[A, B]$ is controllable;

(ii) $\operatorname{rank}(B, AB, \cdots, A^{n-1}B) = n$;

(iii) $\operatorname{rank}(B, \lambda I - A) = n, \quad \forall \lambda \in \mathbb{C}$.

Example 1. For

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t),$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus,

$$(B, AB) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has rank 2. The system is controllable.

Example 2. For

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t),$$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus,

$$(B, AB) = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

has rank 1. The system is not controllable.

There are many extensions:

- Nonlinear systems (robot motion planning)
- Infinite dimensional systems (PDEs)
- Stochastic systems (option pricing)

4. An LQ Problem

Consider control system:

$$\dot{X}(t) = aX(t) + bu(t), \quad t \in [0, T], \qquad X(0) = X_0, \tag{4.1}$$

with cost functional

$$J(u(\cdot)) = \int_0^T \Big[|X(t)|^2 + |u(t)|^2\Big] dt + |X(T)|^2. \tag{4.2}$$

Basic Problems:

- Existence of optimal controls.
- Characterization of optimal controls.
  * Necessary conditions
  * Sufficient conditions

Recall: Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable.

• If $f(x)$ attends a local minimum at x_0, then $f'(x_0) = 0$.
(Necessary condition)

• If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a local minimum.
(Sufficient condition)

The above are obtained by the so-called *variational method*.

Theorem (Kalman). There exists a unique optimal control for the LQ problem associated with (4.1)–(4.2). Moreover, the optimal control admits the following representation:

$$u(t) = -bP(t)X(t), \qquad t \in [0, T], \tag{4.3}$$

where, $P(t)$ is the solution to the following Riccati equation:

$$\begin{cases} \dot{P}(t) + 2aP(t) - b^2P(t)^2 + 1 = 0, & t \in [0, T], \\ P(T) = 1. \end{cases} \tag{4.4}$$

Representation (4.3) is a *state-feedback* form.

Many extensions:

- Nonlinear cases:
  Pontryagin's maximum principle,
  Bellman's dynamic programming method and HJB equations
- Infinite dimensional systems
  Image processing
  Diffusion-reaction
  Mathematical Biology
- Stochastic systems
  Backward stochastic differential equations,
  Nonlinear probability theory

5. Some Accessible Problems

1. Controllability with initial state on M_0 and terminal state on M_1, $M_0.M_1 \subseteq \mathbb{R}^n$.

2. Controllability with control constraints, e.g., $u(t) \in U$, U is a subspace, a cone, a convex set, etc.

3. LQ problem with controllability requirement, e.g., in addition to the usual LQ problem, we need $X(0) \in M_0$, and $X(T) \in M_1$.

Many similar problems, as well as some application problems in mathematical finance, image processing,...

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Thank You!