Finite Difference Approximation of the Mumford-Shah Functional

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Abstract

We study the pointwise convergence and the Γ-convergence of a family of nonlocal functionals defined in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ to a local functional $\mathcal{F}(u)$ that depends on the gradient of u and on the set of discontinuity points of u. We apply this result to approximate a minimum problem introduced by Mumford and Shah to study edge detection in computer vision theory. © 1998 John Wiley & Sons, Inc.

1 Introduction

Many problems of visual reconstruction (see [7, 16, 17]) and mathematical physics (see [18]) are modeled by free discontinuity problems. Roughly speaking, a free discontinuity problem in an open set $\Omega \subseteq \mathbb{R}^n$ is a problem in which one of the unknowns is a pair (u, K) where $K \subseteq \mathbb{R}^n$ is a closed set and u is a regular function defined in $\Omega \backslash K$.

The prototype of this kind of problems is the variational problem

$$\min\{G(u,K) : K \subseteq \mathbb{R}^n \text{ closed set, } u \in C^1(\Omega\backslash K)\}\,, \tag{1.1}$$

where G is defined for every closed set $K \subseteq \mathbb{R}^n$ and every $u \in C^1(\Omega\backslash K)$ by

$$G(u,K) = \int_{\Omega\backslash K} |\nabla u(x)|^2\,dx + \int_{\Omega\backslash K} |u-g|^p\,dx + \mathcal{H}^{n-1}(\Omega \cap K)\,.$$

Here $1 \le p < +\infty$, $g \in L^p(\Omega) \cap L^\infty(\Omega)$, and $\mathcal{H}^{n-1}$ is the $(n-1)$–dimensional Hausdorff measure.

This variational problem, which was solved by E. De Giorgi, M. Carriero, and A. Leaci in [12], comes from a two-dimensional problem of edge detection in computer vision theory posed by D. Mumford and J. Shah in [17].

Many free discontinuity problems have a natural weak formulation in the space $\mathbf{SBV}(\Omega)$ of special functions with bounded variation or in the larger space $\mathbf{GSBV}(\Omega)$ (see Section 2 for precise definitions). Indeed, for each $u \in \mathbf{GSBV}(\Omega)$ one can define, in a measure-theoretic sense, the approximate gradient $\nabla u(x)$ and the discontinuity set S_u, which turns out to be $(n-1)$–dimensional. The functional

$$\mathcal{F}(u) = \int_\Omega |\nabla u(x)|^2\,dx + \mathcal{H}^{n-1}(S_u \cap \Omega)\,, \quad u \in \mathbf{GSBV}(\Omega)\,,$$

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corresponds in the weak formulation to the "principal part" of $G(u, K)$, i.e., to the terms involving the gradient and the discontinuity set of u.

By the semicontinuity and compactness theorem in $\mathbf{SBV}$ proved by Ambrosio in [2] (cf. also [4] and [1]), variational problems involving $\mathcal{F}$ or more general functionals defined in $\mathbf{SBV}$ or $\mathbf{GSBV}$ can be solved using the direct methods of the calculus of variations.

Moreover, the minimizers of (1.1) can be recovered from minimizers of

$$\mathcal{F}(u) + \int_\Omega |u - g|^p \, dx, \quad u \in \mathbf{GSBV}(\Omega)\,, \tag{1.2}$$

and vice versa (cf. [12]), and the minimum values are equal.

The research on this subject has developed along several directions, among them

- giving nontrivial examples of minimizers for (1.1) or (1.2),
- providing numerical algorithms to approximate such minimizers, and
- finding a reasonable definition of gradient flow associated with $\mathcal{F}$.

A natural approach to these problems is to approximate $\mathcal{F}$ by functionals $\mathcal{F}_\varepsilon$ defined in better spaces, e.g., Sobolev spaces or finite-dimensional vector spaces. These functionals $\{\mathcal{F}_\varepsilon\}$ should converge to $\mathcal{F}$ in the sense of Γ-convergence (see Section 2 for the definition), since this notion is stable under continuous perturbations and guarantees that any limit point of minimizers of $\mathcal{F}_\varepsilon$ is a minimizer for $\mathcal{F}$. Moreover, one can hope to define the gradient flow associated to $\mathcal{F}$ as the limit of the gradient flows associated to $\mathcal{F}_\varepsilon$ (if this limit exists, of course!).

It is easy to see (cf. [8]) that $\mathcal{F}$ *cannot* be approximated in the sense of Γ-convergence by local integral functionals like

$$\int_\Omega f_\varepsilon(\nabla u(x))\, dx\,,$$

defined in the Sobolev space $W^{1,2}(\Omega)$.

This difficulty has been overcome in different ways: Ambrosio and Tortorelli [5, 6] introduced an auxiliary variable, Braides and Dal Maso [8] used nonlocal functionals depending on the average of the gradient in small balls, and Chambolle [9] proved that the discrete weak membrane model of Blake and Zisserman [7] Γ-converges to the functional $\mathcal{F}$ in the case where $n = 1$ and $\Omega = [0, 1]$.

The discrete functionals introduced by Blake and Zisserman generalize to any space dimension, but unfortunately in $\dim \geq 2$ the Γ-limit is not invariant by rigid motions; hence it does not coincide with $\mathcal{F}$.

To overcome this difficulty, De Giorgi proposed the following nonlocal functionals:

$$\mathcal{F}_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\mathbb{R}^n\times\mathbb{R}^n} \arctan\left(\frac{(u(x+\varepsilon\xi)-u(x))^2}{\varepsilon}\right) e^{-|\xi|^2}\,d\xi\,dx\,, \tag{1.3}$$

defined for every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and every $\varepsilon > 0$. He conjectured that the family $\{\mathcal{F}_\varepsilon\}$ should Γ-converge, up to some constants, to $\mathcal{F}$ as $\varepsilon \to 0^+$.

In this paper we prove this conjecture. To be precise, let us set

$$MS_{\lambda,\mu}(u,\Omega) = \begin{cases} \lambda\int_\Omega |\nabla u(x)|^2\,dx + \mu\mathcal{H}^{n-1}(S_u) & \text{if } u \in \mathbf{GSBV}(\Omega)\,, \\ +\infty & \text{if } u \in L^1_{\mathrm{loc}}(\Omega)\backslash\mathbf{GSBV}(\Omega)\,. \end{cases} \tag{1.4}$$

Then we prove that

1. $\mathcal{F}_\varepsilon(u) \leq \frac{\pi^{n/2}}{2} MS_{1,\sqrt{\pi}}(u,\mathbb{R}^n)$ for all $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,
2. $\{\mathcal{F}_\varepsilon(u)\}$ pointwise converges to $\frac{\pi^{n/2}}{2} MS_{1,\sqrt{\pi}}(u,\mathbb{R}^n)$, and
3. $\frac{\pi^{n/2}}{2} MS_{1,\sqrt{\pi}}(u,\mathbb{R}^n)$ is the Γ-limit of $\{\mathcal{F}_\varepsilon(u)\}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

The proof of these results is based on an integral-geometric approach in such a way that it is enough to prove (1), (2), and (3) for a simpler family of one-dimensional functionals $\{F_\varepsilon\}$ (cf. Theorem 3.4).

Moreover, we prove the following compactness result:

4. If $\sup_{\varepsilon>0}\{\mathcal{F}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty\} < +\infty$, then the family $\{u_\varepsilon\}$ is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Thanks to (4) and the variational properties of Γ-convergence, we can approximate the minimum problem (1.2) in the following way:

5. For all $\varepsilon > 0$ there exists a minimum point u_ε of the functional
$$\mathcal{F}_\varepsilon(u) + \int_{\mathbb{R}^n} |u-g|^p\,dx$$
in a suitable class of functions of bounded variation (see Theorem 6.1 for the details), and $\{u_\varepsilon\}$ converges, up to subsequences, to a minimum point of (1.2).

Since the aim of this paper is to prove De Giorgi's conjecture, we will focus our attention on the functionals $\mathcal{F}_\varepsilon$ defined in (1.3); however, the same techniques work for a large class of functionals (see the discussion in Section 7).

This paper is organized as follows: In Section 2 we give notation and preliminaries; in Section 3 we study the convergence of a family of one-dimensional functionals F_ε closely related to $\mathcal{F}_\varepsilon$ (Theorem 3.4), and we prove De Giorgi's conjecture in $\dim = 1$ (Theorem 3.6); in Section 4 we prove De Giorgi's conjecture in $\dim \geq 2$ (Theorem 4.4); in Section 5 we prove the compactness result (4); in Section 6 we prove the approximation result (5); and in Section 7 we extend the results of the previous sections to functionals more general then $\mathcal{F}_\varepsilon$.

2 Preliminaries

In this section we fix notation and we recall basic definitions from the theory of $\mathbf{SBV}$ functions and Γ-convergence.

For all $\alpha \in \mathbb{R}$ the integer part of α is denoted by $[\alpha] = \sup\{z \in \mathbb{Z} : z \leq \alpha\}$. Given $x, y \in \mathbb{R}^n$, their scalar product is denoted by $\langle x, y \rangle$, and the Euclidean norm of x by $|x|$. Given $a, b \in \mathbb{R}$, the closed and left-open interval in $\mathbb{R}$ with extremes a and b are denoted by $[a, b]$ and $]a, b]$, respectively. The maximum and the minimum of $\{a, b\}$ are denoted by $a \vee b$ and $a \wedge b$, respectively. Given $A, B \subseteq \mathbb{R}^n$, we write $A \Subset B$ if the closure of A is compact and contained in B.

The Lebesgue measure and the $(n-1)$–dimensional Hausdorff measure of a set $B \subseteq \mathbb{R}^n$ are denoted by $|B|$ and $\mathcal{H}^{n-1}(B)$, respectively. We use standard notation for the Banach spaces $L^p(\mathbb{R}^n)$ and for the metrizable spaces $L^p_{\text{loc}}(\mathbb{R}^n)$. All the functionals introduced in this paper, and also all the operations of $\lim$, $\liminf$, and $\limsup$, are intended with range in the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

2.1 Special Functions of Bounded Variation

For the general theory of functions with bounded variation, we refer to [14, 19]; here we just recall some definitions and some basic results.

Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let $u : \Omega \to \mathbb{R}$ be a measurable function, and let $x \in \Omega$. We say that $z \in \overline{\mathbb{R}}$ is the approximate limit of u at x, and we write

$$z = \text{ap} - \lim_{y \to x} u(y)$$

if for every neighborhood U of z in $\overline{\mathbb{R}}$ we have that

$$\lim_{\rho\to 0} \frac{1}{\rho^n} |\{y \in \Omega : |y-x| < \rho,\ u(y) \notin U\}| = 0\,.$$

If $z \in \mathbb{R}$ we say that x is a Lebesgue point of u. We say that the vector $\nabla u(x) \in \mathbb{R}^n$ is the approximate gradient of u at x if

$$\mathrm{ap} - \lim_{y\to x} \frac{u(y) - u(x) - \langle \nabla u(x), y - x\rangle}{|y-x|} = 0\,.$$

We denote by S_u the discontinuity set of u, i.e.,

$$S_u := \{x \in \Omega : \mathrm{ap} - \lim_{y\to x} u(y) \text{ does not exist}\}.$$

We say that u is a *function of bounded variation* in Ω, and we write $u \in \mathbf{BV}(\Omega)$, if $u \in L^1(\Omega)$ and its distributional derivative is a vector-valued measure Du with finite total variation $|Du|(\Omega)$. If $u \in \mathbf{BV}(\Omega)$, then S_u turns out to be countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable, i.e.,

$$S_u = N \cup \bigcup_{i\in\mathbb{N}} K_i\,,$$

where $\mathcal{H}^{n-1}(N) = 0$, and each K_i is a compact set contained in a C^1 hypersurface.

Now let us write $Du = D^a u + D^s u$, where $D^a u$ is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$, and $D^s u$ is singular with respect to $\mathcal{L}^n$. The density of $D^a u$ with respect to $\mathcal{L}^n$ coincides almost everywhere with the approximate gradient $\nabla u(x)$. Moreover, we call the restriction $D^j u$ of $D^s u$ to S_u the "jump part" of Du and the restriction $D^c u$ of $D^s u$ to $\Omega \backslash S_u$ the "Cantor part" of Du.

With these notations we have the following decomposition:

$$Du = D^a u + D^j u + D^c u\,.$$

The reader interested in the structure of $D^a u$, $D^j u$, and $D^c u$ is referred to [2].

DEFINITION 2.1 Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $u \in \mathbf{BV}(\Omega)$. We say that u is a *special function of bounded variation*, and we write $u \in \mathbf{SBV}(\Omega)$, if $D^c u = 0$.

Sometimes it is useful to consider the following larger class.

DEFINITION 2.2 Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $u : \Omega \longrightarrow \mathbb{R}$ be a measurable function. We say that u is a *generalized special function of bounded variation*, and we write $u \in \mathbf{GSBV}(\Omega)$, if the truncations $u_k = (u \wedge k) \vee (-k)$ belong to $\mathbf{SBV}(\Omega')$ for every $k > 0$, and every open set $\Omega' \Subset \Omega$.

Every $u \in \mathbf{GSBV}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ has an approximate gradient $\nabla u(x)$ for a.e. $x \in \Omega$, and a countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable discontinuity set S_u. Moreover,

$$\nabla u_k \longrightarrow \nabla u \quad \text{a.e. in } \Omega\,, \qquad \mathcal{H}^{n-1}(S_{u_k}) \longrightarrow \mathcal{H}^{n-1}(S_u)\,,$$

as $k \to \infty$.

The spaces $\mathbf{SBV}(\Omega)$ and $\mathbf{GSBV}(\Omega)$ have been introduced by De Giorgi and Ambrosio in [11] and have been studied in [3]. In this paper we use the following semicontinuity result:

THEOREM 2.3 *For all $\lambda > 0$ and $\mu > 0$, the functional $u \to MS_{\lambda,\mu}(u, \mathbb{R})$ defined in* (1.4) *is lower-semicontinuous in $L^1_{\mathrm{loc}}(\mathbb{R})$.*

This result can be proved directly or simply deduced from the general theory of [3].

REMARK 2.4 In this paper we need the lower semicontinuity of $MS_{\lambda,\mu}$ only in the one-dimensional case. Since the Γ-limit of any family of functions is lower-semicontinuous, our Γ-convergence results provide an alternative proof of the lower semicontinuity of $MS_{\lambda,\mu}$ in $\dim \geq 2$.

2.2 Γ-Convergence

DEFINITION 2.5 Let X be a metric space, let $\{F_i\}$ be a sequence of functions defined in X with values in $\overline{\mathbb{R}}$, and let $F : X \longrightarrow \overline{\mathbb{R}}$. We say that $\{F_i\}$ Γ-*converges* to F, and we write

$$F(x) = \Gamma - \lim_{i \to \infty} F_i(x)\,,$$

if the following two conditions are satisfied:

(i) for every $x \in X$ and every sequence $\{x_i\}$ converging to x, we have that

$$F(x) \leq \liminf_{i \to \infty} F_i(x_i)\,;$$

(ii) for every $x \in X$, there exists a sequence $\{x_i\}$ converging to x such that

$$F(x) \geq \limsup_{i \to \infty} F_i(x_i)\,.$$

The Γ-limit, when it exists, is unique and stable under subsequences.

REMARK 2.6 The notion of Γ-convergence is stable under continuous perturbations in the following sense: If $\{F_i\}$ Γ-converges to F and $G : X \longrightarrow \mathbb{R}$ is continuous, then $\{F_i + G\}$ Γ-converges to $F + G$.

The following theorem clarifies the variational character of Γ-convergence (cf. [10, 13]):

THEOREM 2.7 *Let $\{F_i\}$ be a sequence of functions defined on a metric space X that Γ-converges to a function F. For every $i \in \mathbb{N}$, let x_i be a minimum point for F_i. If the sequence $\{x_i\}$ converges in X to some x_0, then x_0 is a minimum point of F, and $\{F_i(x_i)\}$ converges to $F(x_0)$.*

Finally, we say that a family $\{F_\varepsilon\}_{\varepsilon>0}$ of functions Γ-converges to F as $\varepsilon \to 0^+$ if $\{F_{\varepsilon_i}\}$ Γ-converges to F for every sequence $\{\varepsilon_i\} \to 0^+$.

3 The One-Dimensional Case

In this section we consider the functional

$$F_\varepsilon(u,\Omega) = \frac{1}{\varepsilon}\int_\Omega \arctan\left(\frac{(u(x+\varepsilon)-u(x))^2}{\varepsilon}\right) dx\,, \tag{3.1}$$

defined for every $\varepsilon > 0$, $u \in L^1_{\text{loc}}(\mathbb{R})$, and every measurable set $\Omega \subseteq \mathbb{R}$, with values in $\mathbb{R} \cup \{+\infty\}$. We prove that

$$F_\varepsilon(u,\mathbb{R}) \leq MS_{1,\frac{\pi}{2}}(u,\mathbb{R})$$

for all $\varepsilon > 0$, and all $u \in L^1_{\text{loc}}(\mathbb{R})$ and that $MS_{1,\frac{\pi}{2}}(u,\mathbb{R})$ is the pointwise limit and the Γ-limit of $F_\varepsilon(u,\mathbb{R})$ (Theorem 3.4). This result is the basic tool to study the convergence of $\{\mathcal{F}_\varepsilon\}$, both in the one-dimensional case (Theorem 3.6) and in the general case (Theorem 4.4).

Before we state the precise results for $\{F_\varepsilon\}$, we need some technical lemmata.

LEMMA 3.1 *Let $\alpha \geq 0$ and $\beta > 0$. Then for all $\varepsilon \in]0,\beta]$ there exists*

$$\Psi(\varepsilon,\alpha,\beta) = \min\left\{\sum_{i=1}^{N_\varepsilon} \arctan\left(\frac{x_i^2}{\varepsilon}\right) : \sum_{i=1}^{N_\varepsilon} |x_i| \geq \alpha,\ N_\varepsilon = \left[\frac{\beta}{\varepsilon}\right]\right\}. \tag{3.2}$$

Moreover,

$$\lim_{\varepsilon\to 0^+} \Psi(\varepsilon,\alpha,\beta) = \min\left\{\frac{\alpha^2}{\beta}, \frac{\pi}{2}\right\}$$

for all $\alpha \geq 0$ and $\beta > 0$.

PROOF: *Step* 1. The minimum problem (3.2) has at least one solution, since we can restrict to the compact set

$$\left\{(x_1, \dots, x_{N_\varepsilon}) \in [0,\alpha]^{N_\varepsilon} : \sum_{i=1}^{N_\varepsilon} x_i = \alpha\right\}.$$

Therefore $\Psi(\varepsilon, \alpha, \beta)$ is well-defined.

Step 2. Let us show that

$$\limsup_{\varepsilon\to 0^+} \Psi(\varepsilon,\alpha,\beta) \leq \min\left\{\frac{\alpha^2}{\beta}, \frac{\pi}{2}\right\}. \tag{3.3}$$

To this end, setting $x_1 = \alpha$, $x_2 = \cdots = x_{N_\varepsilon} = 0$ in (3.2), we obtain

$$\Psi(\varepsilon,\alpha,\beta) \leq \arctan\left(\frac{\alpha^2}{\varepsilon}\right), \tag{3.4}$$

while setting $x_1 = \cdots = x_{N_\varepsilon} = \alpha/N_\varepsilon$ yields

$$\Psi(\varepsilon,\alpha,\beta) \leq N_\varepsilon \arctan\left(\frac{\alpha^2}{\varepsilon N_\varepsilon^2}\right). \tag{3.5}$$

Recalling that $N_\varepsilon = [\frac{\beta}{\varepsilon}]$, and taking the $\limsup$ in (3.4) and (3.5), inequality (3.3) follows.

Step 3. Let us show that

$$\liminf_{\varepsilon\to 0^+} \Psi(\varepsilon,\alpha,\beta) \geq \min\left\{\frac{\alpha^2}{\beta}, \frac{\pi}{2}\right\}. \tag{3.6}$$

Let $\{\varepsilon_n\} \to 0^+$ be a sequence such that

$$\liminf_{\varepsilon\to 0^+} \Psi(\varepsilon,\alpha,\beta) = \lim_{n\to\infty} \Psi(\varepsilon_n,\alpha,\beta),$$

and, for each ε_n, let $x_{n,1} \geq \cdots \geq x_{n,N_{\varepsilon_n}}$ be a minimizer for (3.2). Since the function $r \to \arctan(r^2/\varepsilon_n)$ is convex in $[0, 3^{-1/4}\sqrt{\varepsilon_n}]$ and concave in $[3^{-1/4}\sqrt{\varepsilon_n}, +\infty[$, then only $x_{n,1}$ can be greater then $3^{-1/4}\sqrt{\varepsilon_n}$, and all the $x_{n,i}$'s in the convexity zone are necessarily equal. This implies that there are only two possibilities:

(P1) $x_{n,1} = \cdots = x_{n,N_{\varepsilon_n}} = \alpha/N_{\varepsilon_n}$. In this case

$$\Psi(\varepsilon_n, \alpha, \beta) = N_{\varepsilon_n} \arctan\left(\frac{\alpha^2}{\varepsilon_n N_{\varepsilon_n}^2}\right). \tag{3.7}$$

(P2) $x_{n,1} > 3^{-1/4}\sqrt{\varepsilon_n}$, and $x_{n,2} = \cdots = x_{n,N_{\varepsilon_n}} = (\alpha - x_{n,1})/(N_{\varepsilon_n} - 1)$. In this case

$$\Psi(\varepsilon_n, \alpha, \beta) = \arctan\left(\frac{(x_{n,1})^2}{\varepsilon_n}\right) + (N_{\varepsilon_n} - 1)\arctan\left(\frac{(\alpha - x_{n,1})^2}{\varepsilon_n (N_{\varepsilon_n} - 1)^2}\right). \tag{3.8}$$

Up to subsequences, we can assume that either (P1) or (P2) holds for all $n \in \mathbb{N}$. In the first case, by (3.7) we have that

$$\lim_{n\to\infty} \Psi(\varepsilon_n, \alpha, \beta) = \frac{\alpha^2}{\beta};$$

hence (3.6) is satisfied.

In the second case, up to subsequences, we can assume that there exists

$$l = \lim_{n\to\infty} \frac{x_{n,1}}{\sqrt{\varepsilon_n}} \in \mathbb{R} \cup \{+\infty\}.$$

If $l = +\infty$, then the first summand in (3.8) tends to $\frac{\pi}{2}$. If $l \in \mathbb{R}$, then $x_{n,1} \to 0$; hence the second summand in (3.8) tends to α^2/β. In both cases (3.6) is satisfied.

The proof is thus complete. ∎

LEMMA 3.2 *Let $I = [a, b]$ be an interval, let $\{u_\varepsilon\} \subseteq L^1_{\mathrm{loc}}(\mathbb{R})$, and let $u \in L^1_{\mathrm{loc}}(\mathbb{R})$. Let us assume the following:*

(i) *$u_\varepsilon \longrightarrow u$ in $L^1_{\mathrm{loc}}(\mathbb{R})$;*

(ii) *a and b are Lebesgue points of u.*

Then

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, I) \ge \min\left\{\frac{\pi}{2}, \frac{(u(b) - u(a))^2}{b - a}\right\}. \tag{3.9}$$

PROOF: *Step* 1. Let us set $J := |u(b) - u(a)|$. If $J = 0$, there is nothing to prove. We can therefore assume $J > 0$. Let $\{\varepsilon_n\} \longrightarrow 0^+$ be a sequence such that

$$\liminf_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon, I) = \lim_{n\to\infty} F_{\varepsilon_n}(u_{\varepsilon_n}, I).$$

Up to subsequences we can also assume that

$$u_{\varepsilon_n}(x) \longrightarrow u(x) \quad \text{for a.e. } x \in I\,.$$

Now let us fix $\eta \in]0, J]$, let us set $N_{\varepsilon_n} = [|I|/\varepsilon_n]$, and let us define

$$C_n := \left\{ x \in [a, a+\varepsilon_n] : \sum_{k=1}^{N_{\varepsilon_n}} |u_{\varepsilon_n}(x + k\varepsilon_n) - u_{\varepsilon_n}(x + (k-1)\varepsilon_n)| \geq J - \eta \right\}.$$

Step 2. Let us show that

$$\lim_{n\to\infty} \frac{|C_n|}{\varepsilon_n} = 1\,. \tag{3.10}$$

To this end, for all $\delta > 0$ let us set

$$A_\delta := \left\{ x \in [a, a+\delta] : |u(x) - u(a)| < \frac{\eta}{4} \right\},$$
$$B_\delta := \left\{ x \in [b-\delta, b] : |u(x) - u(b)| < \frac{\eta}{4} \right\}.$$

By hypothesis (ii) we have that

$$\lim_{\delta\to 0} \frac{|A_\delta|}{\delta} = \lim_{\delta\to 0} \frac{|B_\delta|}{\delta} = 1\,. \tag{3.11}$$

Moreover, by the Severini-Egorov theorem, there exists $I_\delta \subseteq I$ such that $|I \backslash I_\delta| < \delta^2$ and $u_{\varepsilon_n} \longrightarrow u$ uniformly on I_δ. Therefore, there exists $\overline{n} = \overline{n}(\eta, \delta, I_\delta)$ such that

$$\begin{gathered} n \geq \overline{n}\,, \qquad x \in A_\delta \cap I_\delta\,, \\ y \in B_\delta \cap I_\delta \Longrightarrow |u_{\varepsilon_n}(x) - u_{\varepsilon_n}(y)| \geq J - \eta\,. \end{gathered} \tag{3.12}$$

Moreover,

$$|A_\delta \cap I_\delta| \geq |A_\delta| - \delta^2\,, \quad |B_\delta \cap I_\delta| \geq |B_\delta| - \delta^2\,. \tag{3.13}$$

Now let us set

$$K_n^a := \{k \in \mathbb{N} : [a + k\varepsilon_n, a + (k+1)\varepsilon_n] \subseteq [a, a+\delta]\}\,,$$
$$K_n^b := \{k \in \mathbb{N} : [a + k\varepsilon_n, a + (k+1)\varepsilon_n] \subseteq [b-\delta, b]\}\,.$$

It is easy to see that

$$\sharp K_n^a = [\delta/\varepsilon_n]\,, \qquad \sharp K_n^b \geq ([\delta/\varepsilon_n] - 1)\,, \tag{3.14}$$

where $\sharp S$ denotes the number of elements of the set S.

Finally, let us set

$$A_\delta^n := \{x \in [a, a+\varepsilon_n] : x + k\varepsilon_n \notin A_\delta \cap I_\delta,\ \forall k \in K_n^a\}\,, \tag{3.15}$$
$$B_\delta^n := \{x \in [a, a+\varepsilon_n] : x + k\varepsilon_n \notin B_\delta \cap I_\delta,\ \forall k \in K_n^b\}\,. \tag{3.16}$$

By (3.15), (3.16), and (3.12) it follows that

$$C_n \supseteq [a, a+\varepsilon_n] \backslash (A_\delta^n \cup B_\delta^n)\,, \quad \forall \delta > 0\,,\ \forall n \geq \overline{n}(\eta, \delta, I_\delta)\,. \tag{3.17}$$

Moreover, since

$$\bigcup_{k \in K_n^a} \{A_\delta^n + k\varepsilon_n\} \subseteq [a, a+\delta] \backslash (A_\delta \cap I_\delta)\,,$$
$$\bigcup_{k \in K_n^b} \{B_\delta^n + k\varepsilon_n\} \subseteq [b-\delta, b] \backslash (B_\delta \cap I_\delta)\,,$$

and the above unions are disjoint, by (3.13) and (3.14) it follows that

$$|A_\delta^n| \leq (\delta - |A_\delta| + \delta^2) \left[\frac{\delta}{\varepsilon_n}\right]^{-1}\,,$$
$$|B_\delta^n| \leq (\delta - |B_\delta| + \delta^2) \left(\left[\frac{\delta}{\varepsilon_n}\right] - 1\right)^{-1}\,;$$

hence by (3.17)

$$|C_n| \geq \varepsilon_n - (\delta - |A_\delta| + \delta^2) \left[\frac{\delta}{\varepsilon_n}\right]^{-1} - (\delta - |B_\delta| + \delta^2) \left(\left[\frac{\delta}{\varepsilon_n}\right] - 1\right)^{-1}$$

for all $n > \overline{n}$. Dividing by ε_n and taking the limit as $n \to \infty$, we get

$$\liminf_{n \to \infty} \frac{|C_n|}{\varepsilon_n} \;\geq\; 1 - \frac{1}{\delta}\,(2\delta + 2\delta^2 - |A_\delta| - |B_\delta|)\,.$$

Because δ is arbitrary, we take the limit $\delta \to 0^+$ and by (3.11) we obtain

$$\liminf_{n \to \infty} \frac{|C_n|}{\varepsilon_n} \geq 1\,.$$

Since, of course, $|C_n| \leq \varepsilon_n$, equality (3.10) is proved.

Step 3. Let us prove (3.9). By definition of C_n we have that

$$\begin{aligned}
&F_{\varepsilon_n}(u_{\varepsilon_n}, I)\\
&\geq F_{\varepsilon_n}(u_{\varepsilon_n}, [a, a+\varepsilon_n N_{\varepsilon_n}])\\
&= \int_a^{a+\varepsilon_n N_{\varepsilon_n}} \frac{1}{\varepsilon_n} \arctan\left(\frac{(u_{\varepsilon_n}(x+\varepsilon_n) - u_{\varepsilon_n}(x))^2}{\varepsilon_n}\right) dx\\
&= \int_a^{a+\varepsilon_n} \frac{1}{\varepsilon_n} \sum_{k=1}^{N_{\varepsilon_n}} \arctan\left(\frac{(u_{\varepsilon_n}(x+k\varepsilon_n) - u_{\varepsilon_n}(x+(k-1)\varepsilon_n))^2}{\varepsilon_n}\right) dx\\
&\geq \frac{|C_n|}{\varepsilon_n} \Psi(\varepsilon_n, J-\eta, |I|)\,,
\end{aligned}$$

where Ψ is the function introduced in (3.2).

Applying Lemma 3.1 with $\alpha = J - \eta$, $\beta = |I|$, and recalling (3.10), we can conclude that

$$\begin{aligned}
\lim_{n\to\infty} F_{\varepsilon_n}(u_{\varepsilon_n}, I) &\geq \lim_{n\to\infty} \frac{|C_n|}{\varepsilon_n} \Psi(\varepsilon_n, J-\eta, |I|)\\
&\geq \min\left\{\frac{\pi}{2}, \frac{(J-\eta)^2}{|I|}\right\}.
\end{aligned}$$

Since η is arbitrary, (3.9) is proved. ∎

LEMMA 3.3 *Let* $u \in L^\infty(\mathbb{R})$.

Then there exists $a \in \mathbb{R}$ *such that*

(i) $a + q$ *is a Lebesgue point of* u *for every rational number* q,

(ii) *every sequence* $\{u_n\} \subseteq L^\infty(\mathbb{R})$ *that satisfies the following two conditions:*

- $u_n(a + \frac{z}{n}) = u(a + \frac{z}{n})$ *for all* $z \in \mathbb{Z}$,
- *if* $x \in [a + \frac{z}{n}, a + \frac{z+1}{n}]$, *then* $u_n(x)$ *belongs to the interval with endpoints* $u(a + \frac{z}{n})$ *and* $u(a + \frac{z+1}{n})$,

has a subsequence converging to u *in* $L^1_{\text{loc}}(\mathbb{R})$.

PROOF: We claim that (i) and (ii) are satisfied, for example, for a.e. $a \in [0, 1]$. Since (i) is trivially satisfied for a.e. $a \in [0, 1]$, we can focus our attention on condition (ii).

Step 1. Let us fix some notation. For every $n \geq 1$, $z \in \mathbb{Z}$, and $a \in [0,1]$, let us set

$$I_n^z = \left[a + \frac{z}{n}, a + \frac{z+1}{n}\right] \subseteq \mathbb{R}\,;$$
$$v_n^a(x) = u\left(a + \frac{[n(x-a)]}{n}\right), \quad \forall x \in \mathbb{R}\,;$$
$$w_n^a(x) = u\left(a + \frac{[n(x-a)]+1}{n}\right), \quad \forall x \in \mathbb{R}\,.$$

Thus v_n^a is constant on each interval I_n^z and coincides with the value of u in the infimum of I_n^z, while w_n^a coincides in I_n^z with the value of u in the supremum of I_n^z.

Step 2. Let $I \subseteq \mathbb{R}$ be a fixed interval. We show that there exists a sequence $n_k \to +\infty$ such that

$$v_{n_k}^a \longrightarrow u \quad \text{in } L^1(I) \text{ for a.e. } a \in [0,1]. \tag{3.18}$$

To this end, let us introduce the function

$$g_I^n(a) = \int_I |v_n^a(x) - u(x)|\,dx, \quad a \in [0,1]\,.$$

Since

$$\int_0^1 |v_n^a(x) - u(x)|\,da \leq 2\,|I|\,\|u\|_\infty$$

for all $x \in \mathbb{R}$, and

$$\lim_{n\to\infty} \int_0^1 |v_n^a(x) - u(x)|\,da = \lim_{n\to\infty} n \int_{-\frac{1}{n}}^0 |u(x+y) - u(x)|\,dy = 0$$

for every $x \in I$ that is a Lebesgue point of u, by the dominated convergence theorem it follows that

$$\int_I \int_0^1 |v_n^a(x) - u(x)|\,da\,dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty\,.$$

Changing the order of integration, this proves that $\{g_I^n\} \to 0$ in $L^1([0,1])$; hence there exists a subsequence $\{g_I^{n_k}\}$ that converges to zero for a.e. $a \in [0,1]$.

By the definition of g_I^n, this is equivalent to (3.18).

Step 3. Arguing in the same way for w_n^a and using a diagonal argument, we can prove the existence of a sequence $n_k \to +\infty$ such that

$$v_{n_k}^a \longrightarrow u \quad \text{in } L^1_{\text{loc}}(\mathbb{R}) \text{ for a.e. } a \in [0,1]\,, \tag{3.19}$$

$$w_{n_k}^a \longrightarrow u \quad \text{in } L^1_{\text{loc}}(\mathbb{R}) \text{ for a.e. } a \in [0,1]\,. \tag{3.20}$$

Step 4. Let $\{u_n\}$ be any sequence satisfying the two conditions in (ii). Then

$$v_n^a \wedge w_n^a \leq u_n \leq v_n^a \vee w_n^a \,. \tag{3.21}$$

Let $\{n_k\}$ be the sequence of step 3. Thanks to (3.19), (3.20), and (3.21), by a comparison argument it follows that $u_{n_k} \to u$ in $L^1_{\text{loc}}(\mathbb{R})$ for a.e. $a \in [0,1]$.

The proof is thus complete. ∎

We can now state our main result for the functional F_ε.

THEOREM 3.4 *Let F_ε and $MS_{\lambda,\mu}$ be the functionals defined in* (3.1) *and* (1.4)*, respectively. Then, for all $u \in L^1_{\text{loc}}(\mathbb{R})$, we have that*

$$F_\varepsilon(u,\mathbb{R}) \leq MS_{1,\frac{\pi}{2}}(u,\mathbb{R}) \tag{3.22}$$

and

$$\lim_{\varepsilon\to 0^+} F_\varepsilon(u,\mathbb{R}) = MS_{1,\frac{\pi}{2}}(u,\mathbb{R}) \,. \tag{3.23}$$

Moreover,

$$\Gamma - \lim_{\varepsilon\to 0^+} F_\varepsilon(u,\mathbb{R}) = MS_{1,\frac{\pi}{2}}(u,\mathbb{R}) \,, \tag{3.24}$$

where the Γ-limit is computed with respect to the usual topology of $L^1_{\text{loc}}(\mathbb{R})$.

PROOF: *Step* 1. Let us show that (3.22) holds; hence

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u,\mathbb{R}) \leq MS_{1,\frac{\pi}{2}}(u,\mathbb{R}) \,, \quad \forall u \in L^1_{\text{loc}}(\mathbb{R}) \,.$$

To this end, we can clearly assume that $MS_{1,\frac{\pi}{2}}(u,\mathbb{R}) < +\infty$; hence $u \in \mathbf{BV}(I)$ for every $I \Subset \mathbb{R}$. Let us write

$$\begin{aligned} F_\varepsilon(u,\mathbb{R}) = {} & \frac{1}{\varepsilon}\int_{A_\varepsilon} \arctan\left(\frac{(u(x+\varepsilon)-u(x))^2}{\varepsilon}\right) dx \\ & + \frac{1}{\varepsilon}\int_{\mathbb{R}\setminus A_\varepsilon} \arctan\left(\frac{(u(x+\varepsilon)-u(x))^2}{\varepsilon}\right) dx \,, \end{aligned}$$

where

$$A_\varepsilon := \{x \in \mathbb{R} : [x, x+\varepsilon] \cap S_u \neq \varnothing\} \,,$$

and let us estimate separately the two summands. The first one can be trivially estimated by

$$\frac{1}{\varepsilon}\int_{A_\varepsilon}\arctan\left(\frac{(u(x+\varepsilon)-u(x))^2}{\varepsilon}\right)dx \le \frac{\pi}{2}\frac{|A_\varepsilon|}{\varepsilon} \le \frac{\pi}{2}\mathcal{H}^0(S_u)\,. \tag{3.25}$$

In order to estimate the second summand, we note that if $x \notin A_\varepsilon$, then u is absolutely continuous in $[x, x+\varepsilon]$; hence, by Hölder's inequality

$$(u(x+\varepsilon)-u(x))^2 \le \left(\int_0^\varepsilon |\nabla u(x+\tau)|\,d\tau\right)^2 \le \varepsilon\int_0^\varepsilon |\nabla u(x+\tau)|^2\,d\tau\,.$$

Thus, since $\arctan(x) \le x$ for all $x \ge 0$:

$$\begin{aligned}&\frac{1}{\varepsilon}\int_{\mathbb{R}\setminus A_\varepsilon}\arctan\left(\frac{(u(x+\varepsilon)-u(x))^2}{\varepsilon}\right)dx\\ &\le \frac{1}{\varepsilon}\int_{\mathbb{R}\setminus A_\varepsilon}\int_0^\varepsilon |\nabla u(x+\tau)|^2\,d\tau\,dx\\ &\le \frac{1}{\varepsilon}\int_0^\varepsilon d\tau\int_{\mathbb{R}}|\nabla u(x)|^2\,dx = \int_{\mathbb{R}}|\nabla u(x)|^2\,dx\,.\end{aligned}$$

By (3.25) and this last estimate, (3.22) easily follows.

Step 2. In order to complete the proof of pointwise and Γ-convergence, it remains to show that

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n,\mathbb{R}) \ge MS_{1,\frac{\pi}{2}}(u,\mathbb{R}) \tag{3.26}$$

for every sequence $\{\varepsilon_n\} \to 0^+$ and every sequence $\{u_n\} \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R})$.

By a truncation argument, we can assume that $u \in L^\infty(\mathbb{R})$, $\{u_n\} \subseteq L^\infty(\mathbb{R})$, and $\|u_n\|_\infty \le \|u\|_\infty$.

Our aim is to construct a sequence $\{v_j\} \subseteq \mathbf{GSBV}(\mathbb{R})$ such that

$$v_j \longrightarrow u \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R})\,, \tag{3.27}$$

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n,\mathbb{R}) \ge MS_{1,\frac{\pi}{2}}(v_j,\mathbb{R})\,, \quad \forall j \in \mathbb{N}\,. \tag{3.28}$$

By the semicontinuity of $MS_{1,\frac{\pi}{2}}$ (Theorem 2.3), this will imply (3.26).

Let us assume that $a \in \mathbb{R}$ satisfies conditions (i) and (ii) of Lemma 3.3, and let us denote by I_j^z the interval $[a+\frac{z}{j}, a+\frac{z+1}{j}]$. Let us define v_j in every interval I_j^z in the following way:

- If $j(u(a+\frac{z+1}{j})-u(a+\frac{z}{j}))^2 \le \frac{\pi}{2}$, then v_j is the affine function that coincides with u at the endpoints of I_j^z;

- if $j(u(a+\frac{z+1}{j})-u(a+\frac{z}{j}))^2 > \frac{\pi}{2}$, then v_j is the piecewise constant function that coincides with u at the endpoints of I_j^z and has a unique discontinuity in the medium point of the interval.

It is clear from the definition that $\{v_j\}$ satisfies both assumptions of (ii) of Lemma 3.3; hence, up to subsequences, (3.27) holds.

Moreover, we have that $v_j \in \mathbf{GSBV}(\mathbb{R})$ and

$$MS_{1,\frac{\pi}{2}}(v_j, I_j^z) = \min\left\{\frac{\pi}{2}, j\left(u\left(a+\frac{z+1}{j}\right)-u\left(a+\frac{z}{j}\right)\right)^2\right\}$$

for all $j \in \mathbb{N}$ and $z \in \mathbb{Z}$. Therefore, applying Lemma 3.2 in the interval I_j^z, it follows that

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n, I_j^z) \geq MS_{1,\frac{\pi}{2}}(v_j, I_j^z)\,.$$

Summing over all $z \in \mathbb{Z}$ we obtain (3.28), and this completes the proof. ∎

As an intermediate step between the functional F_ε of (3.1) and the functional $\mathcal{F}_\varepsilon$ of (1.3), let us introduce the functional $F_{\varepsilon,\xi}$ defined, for every $\varepsilon > 0$, $\xi \in \mathbb{R}$, and $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ by

$$F_{\varepsilon,\xi}(u) = \frac{1}{\varepsilon}\int_{\mathbb{R}} \arctan\left(\frac{(u(x+\varepsilon\xi)-u(x))^2}{\varepsilon}\right) dx\,. \tag{3.29}$$

With this notation in the one-dimensional case, we have that

$$\mathcal{F}_\varepsilon(u) = \int_{\mathbb{R}} F_{\varepsilon,\xi}(u)\, e^{-\xi^2}\, d\xi\,, \quad \forall u \in L^1_{\mathrm{loc}}(\mathbb{R})\,. \tag{3.30}$$

In the following proposition, we study the convergence of $F_{\varepsilon,\xi}$.

PROPOSITION 3.5 *Let $\xi \in \mathbb{R}$, and let $F_{\varepsilon,\xi}$ and $MS_{\lambda,\mu}$ be the functionals defined in* (3.29) *and* (1.4)*, respectively. Then, for every $u \in L^1_{\mathrm{loc}}(\mathbb{R})$ we have that*

$$F_{\varepsilon,\xi}(u) \leq MS_{\xi^2,\frac{\pi}{2}|\xi|}(u,\mathbb{R}) \tag{3.31}$$

and

$$\lim_{\varepsilon\to 0^+} F_{\varepsilon,\xi}(u) = MS_{\xi^2,\frac{\pi}{2}|\xi|}(u,\mathbb{R})\,. \tag{3.32}$$

Moreover,

$$\Gamma\!-\!\lim_{\varepsilon\to 0^+} F_{\varepsilon,\xi}(u) = MS_{\xi^2,\frac{\pi}{2}|\xi|}(u,\mathbb{R})\,, \tag{3.33}$$

where the Γ-limit is computed with respect to the usual topology of $L^1_{\mathrm{loc}}(\mathbb{R})$.

PROOF: Let us write $F_{\varepsilon,\xi}$ in terms of the functional F_ε defined in (3.1). If $\xi > 0$, then

$$F_{\varepsilon,\xi}(u) = \xi\, F_{\varepsilon\xi}(\xi^{1/2}u, \mathbb{R})\,.$$

If $\xi < 0$, then

$$F_{\varepsilon,\xi}(u) = |\xi|\, F_{\varepsilon|\xi|}(|\xi|^{1/2}\overline{u}, \mathbb{R})\,,$$

where $\overline{u}$ is the function defined by $\overline{u}(x) = u(-x)$.

Therefore, since

$$|\xi| MS_{1,\frac{\pi}{2}}\left(|\xi|^{1/2}u, \mathbb{R}\right) = |\xi| MS_{1,\frac{\pi}{2}}\left(|\xi|^{1/2}\overline{u}, \mathbb{R}\right) = MS_{\xi^2,\frac{\pi}{2}|\xi|}(u, \mathbb{R})\,,$$

in both cases (3.31), (3.32), and (3.33) follow from (3.22), (3.23), and (3.24) respectively. ∎

We are now ready to prove De Giorgi's conjecture in the one-dimensional case.

THEOREM 3.6 *Let* $\mathcal{F}_\varepsilon$ *and* $MS_{\lambda,\mu}$ *be the functionals defined in* (1.3) *and* (1.4)*, respectively.*

Then, for all $u \in L^1_{\text{loc}}(\mathbb{R})$ *we have that*

$$\mathcal{F}_\varepsilon(u) \le MS_{\frac{\sqrt{\pi}}{2},\frac{\pi}{2}}(u, \mathbb{R}) \tag{3.34}$$

and

$$\lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) = MS_{\frac{\sqrt{\pi}}{2},\frac{\pi}{2}}(u, \mathbb{R})\,. \tag{3.35}$$

Moreover,

$$\Gamma - \lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) = MS_{\frac{\sqrt{\pi}}{2},\frac{\pi}{2}}(u, \mathbb{R})\,, \tag{3.36}$$

where the Γ*-limit is computed with respect to the usual topology of* $L^1_{\text{loc}}(\mathbb{R})$.

PROOF: Let us prove (3.34). By (3.30) and (3.31)

$$\begin{aligned}\mathcal{F}_\varepsilon(u) &= \int_{\mathbb{R}} e^{-\xi^2}\, F_{\varepsilon,\xi}(u)\, d\xi \;\le\; \int_{\mathbb{R}} e^{-\xi^2}\, MS_{\xi^2,\frac{\pi}{2}|\xi|}(u, \mathbb{R})\, d\xi \\ &= \int_{\mathbb{R}} \xi^2\, e^{-\xi^2} \int_{\mathbb{R}} |\nabla u(x)|^2\, dx\, d\xi + \frac{\pi}{2}\, \mathcal{H}^0(S_u) \int_{\mathbb{R}} |\xi| e^{-\xi^2}\, d\xi \\ &= MS_{\frac{\sqrt{\pi}}{2},\frac{\pi}{2}}(u, \mathbb{R})\,.\end{aligned}$$

In order to complete the proof, it remains to show that

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) \geq MS_{\frac{\sqrt{\pi}}{2},\frac{\pi}{2}}(u,\mathbb{R})$$

for every sequence $\{\varepsilon_n\} \to 0^+$ and every sequence $\{u_n\} \longrightarrow u$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. By (3.30), (3.33), and Fatou's lemma:

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) &= \liminf_{n\to\infty} \int_{\mathbb{R}} e^{-\xi^2}\, F_{\varepsilon_n,\xi}(u_n)\, d\xi \\ &\geq \int_{\mathbb{R}} e^{-\xi^2}\, \liminf_{n\to\infty} F_{\varepsilon_n,\xi}(u_n)\, d\xi \\ &\geq \int_{\mathbb{R}} e^{-\xi^2}\, MS_{\xi^2,\frac{\pi}{2}|\xi|}(u,\mathbb{R})\, d\xi = MS_{\frac{\sqrt{\pi}}{2},\frac{\pi}{2}}(u,\mathbb{R})\,.\end{aligned}$$

This completes the proof. ∎

4 The General Case

In this section we prove the convergence results for $\mathcal{F}_\varepsilon$ in the n-dimensional case. The proof is obtained by reducing to one-dimensional functionals through the use of an integral-geometric technique. To this end let us introduce, in analogy with (3.29), the functionals $F_{\varepsilon,\xi}$ defined for all $\varepsilon > 0$, $\xi \in \mathbb{R}^n$, and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

(4.1) $$F_{\varepsilon,\xi}(u) = \frac{1}{\varepsilon}\int_{\mathbb{R}^n} \arctan\left(\frac{(u(x+\varepsilon\xi)-u(x))^2}{\varepsilon}\right) dx\,.$$

With this notation we have that

(4.2) $$\mathcal{F}_\varepsilon(u) = \int_{\mathbb{R}^n} F_{\varepsilon,\xi}(u)\, e^{-|\xi|^2}\, d\xi\,, \quad \forall u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)\,.$$

Moreover, let us introduce the functional $MS^{\xi}_{\lambda,\mu}$ defined for all $\xi \in \mathbb{R}^n$, $\lambda, \mu > 0$, and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by

(4.3) $$MS^{\xi}_{\lambda,\mu}(u) = \int_{\langle\xi\rangle^\perp} MS_{\lambda,\mu}(u_{\xi,y},\mathbb{R})\, dy\,,$$

where $\langle\xi\rangle^\perp = \{z \in \mathbb{R}^n : \langle\xi, z\rangle = 0\}$ is the orthogonal space to ξ, and for all $\xi \in \mathbb{R}^n$, $y \in \langle\xi\rangle^\perp$, the function $u_{\xi,y} : \mathbb{R} \to \mathbb{R}$ is defined by

(4.4) $$u_{\xi,y}(t) = u(y + t\xi)\,, \quad \forall t \in \mathbb{R}\,.$$

In the next lemma we recall the results we need about one-dimensional sections of **SBV** functions (cf. [2]).

LEMMA 4.1 *Let* $u \in \mathbf{GSBV}(\mathbb{R}^n)$. *Then for all* $\xi \in \mathbb{R}^n$ *we have that* $u_{\xi,y} \in \mathbf{GSBV}(\mathbb{R})$ *for a.e.* $y \in \langle \xi \rangle^{\perp}$, *and moreover,*

$$\nabla u_{\xi,y}(t) = \langle \nabla u(y + t\xi), \xi \rangle \quad \text{for a.e. } t \in \mathbb{R} \tag{4.5}$$

and

$$S_{u_{\xi,y}} = \{t \in \mathbb{R} : y + t\xi \in S_u\}\,. \tag{4.6}$$

Conversely, let $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, *and let* $\{\xi_1, \dots, \xi_n\} \subseteq \mathbb{R}^n$ *be a set of linearly independent vectors. If for some* $\lambda > 0$ *and* $\mu > 0$ *we have that*

$$\int_{\langle \xi_i \rangle^{\perp}} MS_{\lambda,\mu}(u_{\xi_i,y}, \mathbb{R})\, dy < +\infty \tag{4.7}$$

for all $i \in \{1, \dots, n\}$, *then* $u \in \mathbf{GSBV}(\mathbb{R}^n)$.

In the next lemma we write $MS_{\lambda,\mu}$ in terms of $MS^{\xi}_{\lambda,\mu}$. This is the technical core of our integral-geometric approach.

LEMMA 4.2 *Let* $MS^{\xi}_{\lambda,\mu}$ *and* $MS_{\lambda,\mu}$ *be the functionals defined in* (4.3) *and* (1.4), *respectively. Then*

$$\int_{\mathbb{R}^n} |\xi|\, e^{-|\xi|^2} MS^{\xi}_{\lambda,\mu}(u)\, d\xi \;=\; \frac{\pi^{-n/2}}{2}\, MS_{\lambda,\frac{2}{\sqrt{\pi}}\mu}(u, \mathbb{R}^n) \tag{4.8}$$

for all $u \in L^1_{\text{loc}}(\mathbb{R}^n)$.

PROOF: *Step* 1. A simple computation shows that

$$\int_{\mathbb{R}^n} \langle \alpha, \xi \rangle^2\, e^{-|\xi|^2}\, d\xi = \frac{\pi^{n/2}}{2}\, |\alpha|^2 \tag{4.9}$$

for all vectors $\alpha \in \mathbb{R}^n$.

Step 2. Let $n \geq 2$ and let $M \subseteq \mathbb{R}^n$ be a countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable set. For all $\xi \in \mathbb{R}^n$, $y \in \langle \xi \rangle^{\perp}$, let us set

$$M_{\xi,y} = \{t \in \mathbb{R} : y + t\xi \in M\}\,.$$

Then

$$\int_{\mathbb{R}^n} |\xi|\, e^{-|\xi|^2} \left\{ \int_{\langle \xi \rangle^{\perp}} \mathcal{H}^0(M_{\xi,y})\, dy \right\} d\xi = \pi^{\frac{n-1}{2}}\, \mathcal{H}^{n-1}(M)\,. \tag{4.10}$$

Indeed, let us consider the equality

$$\int_{S^{n-1}} \left\{ \int_{\langle v\rangle^\perp} \mathcal{H}^0(M_{v,y})\, dy \right\} dv = 2\omega_{n-1}\, \mathcal{H}^{n-1}(M)\,, \tag{4.11}$$

which may be proved directly using the area-coarea formula (cf. [15, sec. 3.2.22]) or by remarking that the left-hand side of (4.11) is equal to $2\omega_{n-1}$ times the $(n-1)$–dimensional integral-geometric measure of M, which coincides with the $(n-1)$-dimensional Hausdorff measure for ($\mathcal{H}^{n-1}$, $n-1$) rectifiable sets (cf. [15, sec. 3.2.26]).

Using spherical coordinates in (4.10) we therefore have that

$$\begin{aligned} &\int_{\mathbb{R}^n} |\xi|\, e^{-|\xi|^2} \left\{ \int_{\langle\xi\rangle^\perp} \mathcal{H}^0(M_{\xi,y})\, dy \right\} d\xi \\ &\quad = \int_0^\infty \rho^n e^{-\rho^2}\, d\rho \int_{S^{n-1}} \left\{ \int_{\langle v\rangle^\perp} \mathcal{H}^0(M_{v,y})\, dy \right\} dv \\ &\quad = \left(\int_0^\infty \rho^n e^{-\rho^2}\, d\rho \right) 2\omega_{n-1}\, \mathcal{H}^{n-1}(M)\,. \end{aligned}$$

Recalling the explicit expressions for these constants (see, for example, [15, p. 251]), (4.10) is proved.

Step 3. Let us prove (4.8) for $u \in \mathbf{GSBV}(\mathbb{R}^n)$. Thanks to (4.5) and (4.6), for $\xi \neq 0$ we have that

$$\begin{aligned} MS^{\xi}_{\lambda,\mu}(u) &= \int_{\langle\xi\rangle^\perp} dy \int_{\mathbb{R}} \lambda\, \langle \nabla u(y+t\xi), \xi\rangle^2\, dt + \int_{\langle\xi\rangle^\perp} \mu\, \mathcal{H}^0(S_{u_{\xi,y}})\, dy \\ &= \frac{\lambda}{|\xi|} \int_{\mathbb{R}^n} \langle \nabla u(x), \xi\rangle^2\, dx + \mu \int_{\langle\xi\rangle^\perp} \mathcal{H}^0((S_u)_{\xi,y})\, dy\,. \end{aligned}$$

Multiplying by $|\xi|\, e^{-|\xi|^2}$ and integrating in ξ over $\mathbb{R}^n$, by (4.9) and (4.10) we obtain (4.8).

Step 4. Let us prove (4.8) for $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \backslash \mathbf{GSBV}(\mathbb{R}^n)$. In this case $MS^{\xi}_{\lambda,\mu}(u) = +\infty$ for a.e. $\xi \in \mathbb{R}^n$. Indeed, if $MS^{\xi_i}_{\lambda,\mu}(u)$ is finite for a set $\{\xi_1, \dots, \xi_n\}$ of linearly independent vectors, then u satisfies (4.7) and therefore $u \in \mathbf{GSBV}(\mathbb{R}^n)$. This proves that in this case both sides of (4.8) are equal to $+\infty$.

∎

In the following proposition we study the convergence of $F_{\varepsilon,\xi}$.

PROPOSITION 4.3 *Let $\xi \in \mathbb{R}^n$, and let $F_{\varepsilon,\xi}$ and $MS^{\xi}_{\lambda,\mu}$ be the functionals defined in* (4.1) *and* (4.3)*, respectively.*

Then, for every $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ we have that

$$F_{\varepsilon,\xi}(u) \leq |\xi| MS^{\xi}_{1,\frac{\pi}{2}}(u) \tag{4.12}$$

and

$$\lim_{\varepsilon\to 0^+} F_{\varepsilon,\xi}(u) = |\xi| MS^{\xi}_{1,\frac{\pi}{2}}(u)\,. \tag{4.13}$$

Moreover,

$$\Gamma - \lim_{\varepsilon\to 0^+} F_{\varepsilon,\xi}(u) = |\xi| MS^{\xi}_{1,\frac{\pi}{2}}(u)\,, \tag{4.14}$$

where the Γ-limit is computed with respect to the usual topology of $L^1_{\text{loc}}(\mathbb{R}^n)$.

PROOF: Let us first remark that $F_{\varepsilon,\xi}$ may be written as

$$F_{\varepsilon,\xi}(u) = |\xi| \int_{\langle\xi\rangle^\perp} F_\varepsilon(u_{\xi,y}, \mathbb{R})\, dy\,, \tag{4.15}$$

where F_ε is the functional defined in (3.1) and $u_{\xi,y}$ are the one-dimensional sections of u defined in (4.4).

Now let us prove (4.12). By (4.15) and (3.22) we have that

$$\begin{aligned} F_{\varepsilon,\xi}(u) &= |\xi| \int_{\langle\xi\rangle^\perp} F_\varepsilon(u_{\xi,y}, \mathbb{R})\, dy \\ &\leq |\xi| \int_{\langle\xi\rangle^\perp} MS_{1,\frac{\pi}{2}}(u_{\xi,y}, \mathbb{R})\, dy = |\xi| MS^{\xi}_{1,\frac{\pi}{2}}(u)\,. \end{aligned}$$

To prove (4.13) and (4.14), it remains to show that

$$\liminf_{n\to\infty} F_{\varepsilon_n,\xi}(u^n) \geq |\xi| MS^{\xi}_{1,\frac{\pi}{2}}(u)$$

for every sequence $\{\varepsilon_n\} \to 0^+$ and every sequence $\{u^n\} \longrightarrow u$ in $L^1_{\text{loc}}(\mathbb{R}^n)$.

Since

$$u^n_{\xi,y} \longrightarrow u_{\xi,y} \quad \text{in } L^1_{\text{loc}}(\mathbb{R})$$

for a.e. $y \in \langle\xi\rangle^\perp$, by (4.15), (3.24), and Fatou's lemma we have that

$$\begin{aligned} \liminf_{n\to\infty} F_{\varepsilon_n,\xi}(u^n) &= |\xi| \liminf_{n\to\infty} \int_{\langle\xi\rangle^\perp} F_{\varepsilon_n}(u^n_{\xi,y}, \mathbb{R})\, dy \\ &\geq |\xi| \int_{\langle\xi\rangle^\perp} \liminf_{n\to\infty} F_{\varepsilon_n}(u^n_{\xi,y}, \mathbb{R})\, dy \\ &\geq |\xi| \int_{\langle\xi\rangle^\perp} MS_{1,\frac{\pi}{2}}(u_{\xi,y}, \mathbb{R})\, dy = |\xi| MS^{\xi}_{1,\frac{\pi}{2}}(u)\,. \end{aligned}$$

This completes the proof. ∎

In the following theorem we prove De Giorgi's conjecture in dimension $n \geq 2$.

THEOREM 4.4 *Let $\mathcal{F}_\varepsilon$ and $MS_{\lambda,\mu}$ be the functionals defined in* (1.3) *and* (1.4)*, respectively.*

Then, for every $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ we have that

$$\mathcal{F}_\varepsilon(u) \leq \frac{\pi^{n/2}}{2}\, MS_{1,\sqrt{\pi}}(u,\mathbb{R}^n) \tag{4.16}$$

and

$$\lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) = \frac{\pi^{n/2}}{2}\, MS_{1,\sqrt{\pi}}(u,\mathbb{R}^n)\,. \tag{4.17}$$

Moreover,

$$\Gamma - \lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(u) = \frac{\pi^{n/2}}{2} MS_{1,\sqrt{\pi}}(u,\mathbb{R}^n) \tag{4.18}$$

where the Γ-limit is computed with respect to the usual topology of $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

PROOF: Let us prove (4.16). By (4.2), (4.12), and Lemma 4.2, we have that

$$\begin{aligned}\mathcal{F}_\varepsilon(u) = \int_{\mathbb{R}^n} e^{-|\xi|^2}\, F_{\varepsilon,\xi}(u)\, d\xi &\leq \int_{\mathbb{R}^n} |\xi| e^{-|\xi|^2}\, MS^{\xi}_{1,\frac{\pi}{2}}(u)\, d\xi \\ &= \frac{\pi^{n/2}}{2}\, MS_{1,\sqrt{\pi}}(u,\mathbb{R}^n)\,.\end{aligned}$$

To prove (4.17) and (4.18) it remains to show that

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) \geq \frac{\pi^{n/2}}{2}\, MS_{1,\sqrt{\pi}}(u,\mathbb{R}^n)$$

for every sequence $\{\varepsilon_n\} \to 0^+$ and every sequence $\{u_n\} \longrightarrow u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. By (4.2), (4.14), Fatou's lemma, and Lemma 4.2, we have that

$$\begin{aligned}\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) &= \liminf_{n\to\infty} \int_{\mathbb{R}^n} e^{-|\xi|^2}\, F_{\varepsilon_n,\xi}(u_n)\, d\xi \\ &\geq \int_{\mathbb{R}^n} e^{-|\xi|^2}\, \liminf_{n\to\infty} F_{\varepsilon_n,\xi}(u_n)\, d\xi \\ &\geq \int_{\mathbb{R}^n} |\xi|\, e^{-|\xi|^2}\, MS^{\xi}_{1,\frac{\pi}{2}}(u)\, d\xi = \frac{\pi^{n/2}}{2}\, MS_{1,\sqrt{\pi}}(u,\mathbb{R})\,.\end{aligned}$$

This completes the proof. ∎

5 Compactness

The main result of this section is the compactness result of Theorem 5.4. In the sequel we denote by $C^\delta u$ the convolutions

$$C^\delta u(x) = \pi^{-n/2} \int_{\mathbb{R}^n} u(x + \delta\xi)\, e^{-|\xi|^2}\, d\xi\,, \tag{5.1}$$

defined for every $u \in L^\infty(\mathbb{R}^n)$ and every $\delta > 0$.

In a standard way it is possible to show that $C^\delta u \in C^\infty(\mathbb{R}^n)$ and, moreover,

$$\|C^\delta u\|_\infty \le \|u\|_\infty \tag{5.2}$$

and

$$\|\nabla C^\delta u\|_\infty \le c_\delta \,\|u\|_\infty \tag{5.3}$$

where

$$c_\delta = \frac{1}{\delta} \int_{\mathbb{R}^n} \left|\nabla e^{-|\xi|^2}\right| d\xi\,.$$

In the following three lemmata we prove some estimates that will be crucial in the proof of Theorem 5.4.

LEMMA 5.1 *For all $u \in L^\infty(\mathbb{R}^n)$, $\delta > 0$, $\xi \in \mathbb{R}^n$, and $A \Subset \mathbb{R}^n$ we have that*

$$\int_A |u(x + \delta\xi) - u(x)|\, dx \le 8\delta \left(|A|^{1/2} + \|u\|_\infty\right)(1 + F_{\delta,\xi}(u)) \tag{5.4}$$

where $F_{\delta,\xi}$ is the functional introduced in (4.1).

PROOF: Let us set

$$A_\delta = \{x \in A : |u(x + \delta\xi) - u(x)|^2 > \delta\}.$$

Since $x \le \frac{4}{\pi} \arctan x$ for $|x| \le 1$, by Hölder's inequality we have that

$$\begin{aligned}
&\int_{A \setminus A_\delta} |u(x + \delta\xi) - u(x)|\, dx \\
&\quad \le |A|^{1/2} \left\{ \delta \int_{A\setminus A_\delta} \frac{|u(x+\delta\xi) - u(x)|^2}{\delta}\, dx \right\}^{1/2} \\
&\quad \le |A|^{1/2} \left\{ \frac{4\delta^2}{\pi}\, \frac{1}{\delta} \int_{A\setminus A_\delta} \arctan\left(\frac{|u(x+\delta\xi) - u(x)|^2}{\delta} \right) dx \right\}^{1/2} \\
&\quad \le \frac{2}{\sqrt{\pi}}\, \delta\, |A|^{1/2} \{F_{\delta,\xi}(u)\}^{1/2} \le \frac{2}{\sqrt{\pi}}\, \delta\, |A|^{1/2} \{1 + F_{\delta,\xi}(u)\}\,.
\end{aligned} \tag{5.5}$$

Moreover, since $\arctan x \geq \frac{\pi}{4}$ for $x \geq 1$, we have that

$$F_{\delta,\xi}(u) \geq \frac{1}{\delta} \int_{A_\delta} \arctan \left(\frac{|u(x+\delta\xi) - u(x)|^2}{\delta} \right) dx \geq \frac{\pi}{4} \frac{|A_\delta|}{\delta} \, ;$$

hence

$$\int_{A_\delta} |u(x+\delta\xi) - u(x)| \, dx \leq 2 \, \|u\|_\infty |A_\delta| \leq \frac{8}{\pi} \, \delta \, \|u\|_\infty \, F_{\delta,\xi}(u) \, . \tag{5.6}$$

By (5.5) and (5.6), inequality (5.4) follows. ∎

LEMMA 5.2 *For all $u \in L^\infty(\mathbb{R}^n)$, $\delta > 0$, and $A \Subset \mathbb{R}^n$ we have that*

$$\|C^\delta u - u\|_{L^1(A)} \leq 8\delta \left(|A|^{1/2} + \|u\|_\infty \right) (1 + \mathcal{F}_\delta(u)) \, ,$$

where $\mathcal{F}_\delta$ is the functional introduced in (1.3).

PROOF: By (5.1), Lemma 5.1, and (4.2) we have that

$$\begin{aligned}
\|C^\delta u - u\|_{L^1(A)} &\leq \pi^{-n/2} \int_{A \times \mathbb{R}^n} |u(x+\delta\xi) - u(x)| \, e^{-|\xi|^2} \, d\xi \, dx \\
&\leq 8\pi^{-n/2} \, \delta \left(|A|^{1/2} + \|u\|_\infty \right) \int_{\mathbb{R}^n} (1 + F_{\delta,\xi}(u)) \, e^{-|\xi|^2} \, d\xi \\
&\leq 8\delta \left(|A|^{1/2} + \|u\|_\infty \right) (1 + \mathcal{F}_\delta(u)) \, .
\end{aligned}$$

∎

LEMMA 5.3 *For all $\varepsilon > 0$, $k \in \mathbb{N}\backslash\{0\}$, $u \in L^\infty(\mathbb{R}^n)$, we have that*

$$\mathcal{F}_{k\varepsilon}(u) \leq \mathcal{F}_\varepsilon(u) \, . \tag{5.7}$$

PROOF: If $k = 1$, the thesis is trivial. Arguing by induction, we assume that (5.7) holds true for some $k \geq 1$. From the inequality

$$\frac{(A+B)^2}{k+1} \leq A^2 + \frac{B^2}{k} \, , \quad \forall k \geq 1 \, , \ \forall A \in \mathbb{R} \, , \ \forall B \in \mathbb{R} \, ,$$

by the monotonicity and the subadditivity of the function $r \to \arctan r$, it follows that

$$\arctan \left(\frac{(A+B)^2}{k+1} \right) \leq \arctan A^2 + \arctan \frac{B^2}{k} \, .$$

Applying this inequality with

$$A = \frac{u(x + (k+1)\varepsilon\xi) - u(x + k\varepsilon\xi)}{\sqrt{\varepsilon}} \, , \qquad B = \frac{u(x + k\varepsilon\xi) - u(x)}{\sqrt{\varepsilon}} \, ,$$

and dividing by $(k+1)\varepsilon$, it follows that

$$\begin{aligned}\frac{1}{(k+1)\varepsilon}\arctan\left(\frac{(u(x+(k+1)\varepsilon\xi)-u(x))^2}{(k+1)\varepsilon}\right)\\ \le \frac{1}{k+1}\,\frac{1}{\varepsilon}\arctan\left(\frac{(u(x+(k+1)\varepsilon\xi)-u(x+k\varepsilon\xi))^2}{\varepsilon}\right)\\ +\frac{k}{k+1}\,\frac{1}{k\varepsilon}\arctan\left(\frac{(u(x+k\varepsilon\xi)-u(x))^2}{k\varepsilon}\right).\end{aligned}$$

Multiplying by $e^{-|\xi|^2}$ and integrating in x and ξ over $\mathbb{R}^n\times\mathbb{R}^n$, we finally obtain

$$\mathcal{F}_{(k+1)\varepsilon}(u)\;\le\;\frac{1}{k+1}\,\mathcal{F}_\varepsilon(u)\;+\;\frac{k}{k+1}\,\mathcal{F}_{k\varepsilon}(u)\;\le\;\mathcal{F}_\varepsilon(u)\,.$$

This shows that (5.7) also holds true for $(k+1)$ and completes the induction. ∎

We can now prove the following compactness result:

THEOREM 5.4 *Let* $\{u_\varepsilon\}\subseteq L^\infty(\mathbb{R}^n)$. *Let us assume that*

$$\sup_{\varepsilon>0}\{\mathcal{F}_\varepsilon(u_\varepsilon)+\|u_\varepsilon\|_\infty\}<\infty\,. \tag{5.8}$$

Then there exists $\{\varepsilon_k\}\to 0^+$ *and* $u\in\mathbf{GSBV}(\mathbb{R}^n)$ *such that*

$$u_{\varepsilon_k}\longrightarrow u\quad\textit{in } L^1_{\mathrm{loc}}(\mathbb{R}^n)\,.$$

PROOF: We have to show that $\{u_{\varepsilon_j}\}$ is relatively compact in $L^1(A)$ for every sequence $\{\varepsilon_j\}\to 0^+$ and every $A\Subset\mathbb{R}^n$. To this end, it is enough to show that there exists a constant $M>0$ such that for all $\sigma>0$ there exists a set $K_\sigma\subseteq L^1(A)$ such that

1. K_σ is relatively compact in $L^1(A)$, and
2. for all $j\in\mathbb{N}$, there exists $v_j\in K_\sigma$ such that $\|u_{\varepsilon_j}-v_j\|_{L^1(A)}\le M\sigma$.

Let us set

$$K_\sigma=\left\{C^{\varepsilon_j[\frac{\sigma}{\varepsilon_j}]}u_{\varepsilon_j}:\varepsilon_j<\frac{\sigma}{2}\right\}\cup\bigcup_{\varepsilon_j>\frac{\sigma}{2}}\{u_{\varepsilon_j}\}\,.$$

Let us show that K_σ satisfies (1). Since there is only a finite number of $\varepsilon_j > \frac{\sigma}{2}$, it suffices to show that

$$\tilde{K}_\sigma = \left\{ C^{\varepsilon_j [\frac{\sigma}{\varepsilon_j}]} u_{\varepsilon_j} : \varepsilon_j < \frac{\sigma}{2} \right\}$$

is relatively compact in $L^1(A)$. To this end, let us remark that $\tilde{K}_\sigma \subseteq C^1(A)$, and moreover, by (5.2) we have that

$$\left\| C^{\varepsilon_j [\frac{\sigma}{\varepsilon_j}]} u_{\varepsilon_j} \right\|_\infty \leq \| u_{\varepsilon_j} \|_\infty \,, \tag{5.9}$$

and since $\varepsilon_j[\sigma/\varepsilon_j] \geq \frac{\sigma}{2}$, by (5.3) we have that

$$\left\| \nabla C^{\varepsilon_j [\frac{\sigma}{\varepsilon_j}]} u_{\varepsilon_j} \right\|_\infty \leq c_{\frac{\sigma}{2}} \| u_{\varepsilon_j} \|_\infty \,. \tag{5.10}$$

By the Ascoli theorem, $\tilde{K}_\sigma$ is relatively compact in $C^0(A)$, hence in $L^1(A)$.

Let us show that K_σ satisfies (2) with

$$M = \sup_{\varepsilon > 0} \left\{ 8 \left(|A|^{1/2} + \|u_\varepsilon\|_\infty \right) (1 + \mathcal{F}_\varepsilon(u_\varepsilon)) \right\} .$$

If $\varepsilon_j > \frac{\sigma}{2}$, we can simply take $v_j = u_{\varepsilon_j}$. If $\varepsilon_j \leq \frac{\sigma}{2}$, we can take $v_j = C^{\varepsilon_j [\frac{\sigma}{\varepsilon_j}]} u_{\varepsilon_j}$. Indeed, by Lemma 5.2 and Lemma 5.3, we have that

$$\begin{aligned} \left\| C^{\varepsilon_j [\frac{\sigma}{\varepsilon_j}]} u_{\varepsilon_j} - u_{\varepsilon_j} \right\|_{L^1(A)} &\leq 8\sigma \left(|A|^{1/2} + \|u_{\varepsilon_j}\|_\infty \right) \left(1 + \mathcal{F}_{\varepsilon_j [\frac{\sigma}{\varepsilon_j}]}(u_{\varepsilon_j}) \right) \\ &\leq 8\sigma \left(|A|^{1/2} + \|u_{\varepsilon_j}\|_\infty \right) \left(1 + \mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j}) \right) \leq M\sigma \,. \end{aligned}$$

∎

We conclude this section with a remark about the choice of the function spaces.

REMARK 5.5 It is possible to consider the restrictions of $\mathcal{F}_\varepsilon$ and $MS_{\lambda,\mu}$ to the space $L^p_{\text{loc}}(\mathbb{R}^n)$ with $1 \leq p < +\infty$. The results of Sections 3 through 5 may be generalized word by word, but of course the Γ-limit and the compactness are intended to be in $L^p_{\text{loc}}(\mathbb{R}^n)$.

We can also consider the restrictions to $L^p(\mathbb{R}^n)$. In this case the Γ-convergence results (with respect to the strong topology) are still the same, but we can expect compactness only in $L^p_{\text{loc}}(\mathbb{R}^n)$ (just observe that $\mathcal{F}_\varepsilon$ is invariant by translation).

6 Applications to the Mumford-Shah Problem

Thanks to the general properties of Γ-convergence, the results of the previous sections lead to the following approximation of the minimization problem for the Mumford-Shah functional:

THEOREM 6.1 *Let $1 \le p < +\infty$, and let $g \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then for every $\varepsilon > 0$ there exists a solution u_ε to the minimum problem*

(6.1)
$$m_\varepsilon = \min\left\{\mathcal{F}_\varepsilon(u) + \int_{\mathbb{R}^n} |u-g|^p\,dx : u \in \mathbf{BV}(\mathbb{R}^n),\ |Du|(\mathbb{R}^n) \le \frac{1}{\varepsilon}\right\}.$$

Moreover, every sequence $\{u_{\varepsilon_j}\}$ with $\{\varepsilon_j\} \to 0^+$ has a subsequence converging in $L^1_{\rm loc}(\mathbb{R}^n)$ to a solution of the minimum problem

(6.2) $m_0 =$
$$\min_{u\in\mathbf{SBV}(\mathbb{R}^n)}\left\{\frac{\pi^{n/2}}{2}\int_{\mathbb{R}^n}|\nabla u|^2dx + \frac{\pi^{(n+1)/2}}{2}\,\mathcal{H}^{n-1}(S_u) + \int_{\mathbb{R}^n}|u-g|^p\,dx\right\}.$$

Furthermore,
$$\lim_{\varepsilon\to 0} m_\varepsilon = m_0\,.$$

PROOF: *Step* 1. *Existence of minimizers.* By a truncation argument, it is easy to see that we can restrict to the class of all $u \in L^\infty(\mathbb{R}^n)$ such that $\|u\|_\infty \le \|g\|_\infty$. Since the functional

$$\mathcal{F}_\varepsilon(u) + \int_{\mathbb{R}^n}|u-g|^p\,dx$$

is lower-semicontinuous in $L^1_{\rm loc}(\mathbb{R}^n)$ (by Fatou's lemma) and the set

$$\left\{u \in L^\infty(\mathbb{R}^n)\cap\mathbf{BV}(\mathbb{R}^n) : \|u\|_\infty \le \|g\|_\infty,\ |Du|(\mathbb{R}^n) \le \frac{1}{\varepsilon}\right\}$$

is compact in $L^1_{\rm loc}(\mathbb{R}^n)$, the existence of minimizers follows from the direct method of the calculus of variations.

Step 2. *Γ-convergence.* We show that in $L^1_{\rm loc}(\mathbb{R}^n)$ the functional

$$\mathcal{G}_\varepsilon(u) = \begin{cases}\mathcal{F}_\varepsilon(u) + \displaystyle\int_{\mathbb{R}^n}|u-g|^p\,dx & u \in \mathbf{BV}(\mathbb{R}^n),\ |Du|(\mathbb{R}^n) \le \frac{1}{\varepsilon}\\ +\infty & \text{otherwise}\end{cases}$$

Γ-converges to the functional

$$\mathcal{G}(u) = \frac{\pi^{n/2}}{2} \, MS_{1,\sqrt{\pi}}(u, \mathbb{R}^n) + \int_{\mathbb{R}^n} |u - g|^p \, dx \, .$$

The $\liminf$ inequality follows from the Γ-convergence of $\{\mathcal{F}_\varepsilon\}$ and Fatou's lemma. The $\limsup$ inequality follows from the pointwise convergence of $\{\mathcal{F}_\varepsilon\}$ if $u \in \mathbf{SBV}(\mathbb{R}^n)$ and hence, by the usual truncation argument, for all u.

Step 3. *Compactness.* Let $\{u_\varepsilon\}$ be a family of minimizers for (6.1). In step 1 we proved that $\|u_\varepsilon\|_\infty \leq \|g\|_\infty$. Moreover, since in (6.1) we can always take $u = 0$, it follows that $\mathcal{F}_\varepsilon(u_\varepsilon) \leq \|g\|_p^p$, so that

$$\sup_{\varepsilon > 0} \{\mathcal{F}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty\} < \infty \, .$$

By Theorem 5.4 the family $\{u_\varepsilon\}$ is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Step 4. *Conclusion.* By the general properties of Γ-convergence (Theorem 2.7), any limit point of $\{u_\varepsilon\}$ is a minimizer for (6.2) and $m_\varepsilon \to m_0$.

∎

The a priori restriction $u \in \mathbf{BV}(\mathbb{R}^n)$, $|Du|(\mathbb{R}^n) \leq \frac{1}{\varepsilon}$, has been introduced in the minimum problem (6.1) only for technical reasons, i.e., to prove existence of minimizers applying the direct method in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Open problem. Does the problem

$$\min \left\{ \mathcal{F}_\varepsilon(u) + \int_{\mathbb{R}^n} |u - g|^p \, dx : u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \right\} \tag{6.3}$$

have a solution for every $\varepsilon > 0$ and every $g \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$?

We know that the answer is yes if ε is large enough (in this case it is possible to apply the direct method with respect to the weak $*$-topology of $L^\infty(\mathbb{R}^n)$), but, unfortunately, this is not what we need in the approximation to (6.2).

Moreover, looking at the Euler equation for (6.3), one can prove that if (6.3) has a minimizer that is continuous (respectively, differentiable, $\mathbf{BV}$), then g is necessarily continuous (respectively, differentiable, $\mathbf{BV}$). This shows that if $g \notin \mathbf{BV}(\mathbb{R}^n)$, then the minimizers of (6.1) are not minimizers of (6.3).

7 Generalizations

In this section we consider the family of functionals defined by

$$\mathcal{F}_\varepsilon^{f,J}(u,\Omega) = \frac{1}{\varepsilon}\int_{\Omega\times\Omega} f\left(\frac{(u(y)-u(x))^2}{\varepsilon}\right) J_\varepsilon(|y-x|)\,dx\,dy\,, \tag{7.1}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, $u \in L^1_{\text{loc}}(\Omega)$, $\varepsilon > 0$, f and J are given functions, and

$$J_\varepsilon(t) = \frac{1}{\varepsilon^n}\, J\left(\frac{t}{\varepsilon}\right), \quad \forall t \geq 0\,.$$

On the functions f and J we make the following assumptions:

(Hp-f) $f : [0,+\infty[\to [0,+\infty[$ is a continuous, nondecreasing, subadditive function such that $f(0) = 0$, $f(t) > 0$ for every $t > 0$, and there exist the following limits:

$$f'(0) = \lim_{t\to 0^+} \frac{f(t)}{t} > 0\,, \qquad f_\infty = \lim_{t\to\infty} f(t)\,;$$

(Hp-J) $J : [0,+\infty[\to [0,+\infty[$ is measurable, and for every $k \in \mathbb{N}$ the following integrals are finite:

$$j(k) := \int_0^\infty t^k J(t)\,dt\,. \tag{7.2}$$

Moreover, let us assume for simplicity that

(Hp-Ω) the open set Ω is convex.

If $f(t) = \arctan(t)$, $J(t) = e^{-t^2}$, and $\Omega = \mathbb{R}^n$, then the functionals $\mathcal{F}_\varepsilon$ of (1.3) may be written as in (7.1) setting $y = x + \varepsilon\xi$.

All the results of the previous sections (in particular, pointwise convergence, Γ-convergence, compactness, and approximation of the Mumford-Shah problem) may be generalized to the family $\{\mathcal{F}_\varepsilon^{f,J}(u,\Omega)\}$. In this case the pointwise limit and Γ-limit (with respect to the usual topology of $L^1_{\text{loc}}(\Omega)$) is $MS_{\lambda,\mu}(u,\Omega)$ with

$$\lambda = f'(0)\,\omega_n\, j(n+1)\,, \qquad \mu = 2 f_\infty\, \omega_{n-1}\, j(n)\,,$$

where ω_k denotes the Lebesgue measure of the unit sphere in $\mathbb{R}^k$, and $j(k)$ is given by (7.2).

Moreover, if we restrict $\mathcal{F}_\varepsilon^{f,J}(u,\Omega)$ to $L^p(\Omega)$, we can prove a compactness result analogous to Theorem 5.4 with respect to the strong topology of $L^p(\Omega)$ provided that Ω is bounded. Indeed, in this case the family $\{\tilde{u}_\varepsilon\} \subseteq L^p(\mathbb{R}^n)$ obtained by extending the u_ε's to zero in $\mathbb{R}^n \backslash \Omega$ still satisfies (5.8). Therefore $\{\tilde{u}_\varepsilon\}$ is relatively compact in $L^p_{\text{loc}}(\mathbb{R}^n)$ (see Remark 5.5). Since Ω is bounded, $\{u_\varepsilon\}$ is relatively compact in $L^p(\Omega)$.

REMARK 7.1 These results are false for a general open set Ω. For example, let us take $f(t) = \arctan(t)$, $J(t) = e^{-t^2}$, and $\Omega = \mathbb{R}\backslash\{0\}$, and let us consider the function

$$u(t) = \begin{cases} 0 & x < 0\,, \\ 1 & x > 0\,. \end{cases}$$

It is clear that $MS_{\lambda,\mu}(u,\Omega) = 0$ for all λ and μ. On the other hand,

$$\mathcal{F}_\varepsilon^{f,J}(u,\Omega) = \mathcal{F}_\varepsilon^{f,J}(u,\mathbb{R})\,;$$

hence, by Theorem 3.6,

$$\lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon^{f,J}(u,\Omega) = \lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon^{f,J}(u,\mathbb{R}) = MS_{\frac{\sqrt{\pi}}{2},\frac{\pi}{2}}(u,\mathbb{R}) = \frac{\pi}{2}\,.$$

This shows that $MS_{\frac{\sqrt{\pi}}{2},\frac{\pi}{2}}(u,\Omega)$ is not the pointwise limit of $\mathcal{F}_\varepsilon^{f,J}(u,\Omega)$.

The assumptions (Hp-f) and (Hp-Ω) are far from being optimal. In particular, all the convergence, compactness, and approximation results remain true if the monotonicity and subadditivity assumptions on f are dropped (but some proofs become more cumbersome!). The same results also hold true if Ω is not necessarily convex but satisfies the following:

(Hp-Ω)$'$ the open set Ω is bounded and smooth.

Another possibility is to restrict the domain of integration in (7.1) to the set

$$\{(x,y) \in \Omega \times \Omega : \forall t \in [0,1],\ tx + (1-t)y \in \Omega\}\,.$$

Roughly speaking, this means that x and y interact if and only if they see each other. With this choice all the theory works without any restriction on Ω.

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