Variational Method for Image Analysis

Content

- Identify the Solution as the Minimum
- Derivation of the Euler Equations
- Finite Difference Methods for Solving PDEs
- Image Segmentation Through Active Contours
- Denoising
- Optical Flow
- Target Tracking

1 Solution as Minimum

There are many cases where we obtain a solution from a minimization problem. For example, in finding the sparse representation, we used a minimization (i.e., minimum l_1) problem to single out the solution among all possible solutions of an underdetermined linear system. When we face a overdetermined system, say $A\mathbf{x} = \mathbf{b}$ where we do not have any exact solution, we can use a minimization problem to find an approximate solution by minimizing the norm of $A\mathbf{x} - \mathbf{b}$. Indeed, the least squares method is the one when we minimize the l_2 norm of $A\mathbf{x} - \mathbf{b}$:

$$\|A\mathbf{x} - \mathbf{b}\|_2 \to \min$$

Homework. Assume that A is a $m \times n$ matrix with $m > n$ and $\mathbf{b} \in \mathbb{R}^m$. Show that the solution $\mathbf{x}^* \in \mathbb{R}^m$ to

$$\min_{\mathbf{x} \in \mathbb{R}^m} \|A\mathbf{x} - \mathbf{b}\|_2^2$$

satisfies

$$A^t A\mathbf{x} = A^t\mathbf{b}.$$

So, we arrive at some equations from solving a minimization problem. This is the basic idea of variational method.

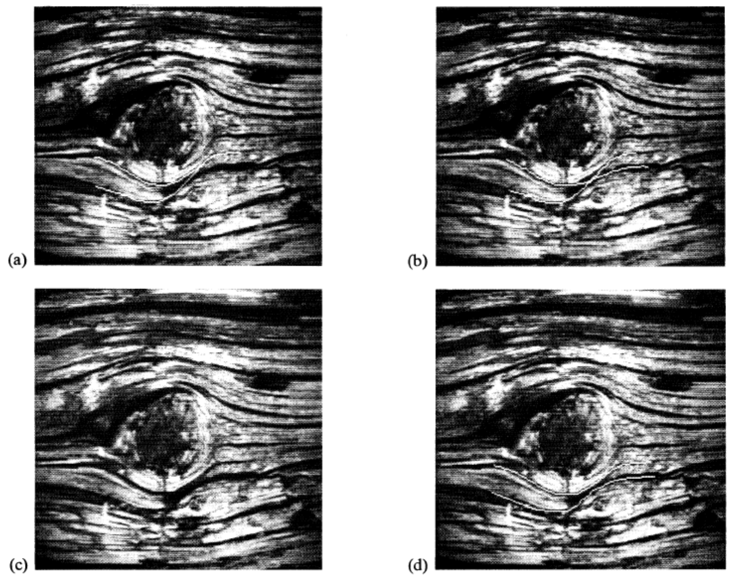

Figure 1: Lower-left corner: Original wood photograph of Brodatz. Other: Three different local minima for the active contour model. (From the paper of Kass, Witkin, Terzopuolos (IJCV, 1988))

2 Snakes: Active Contours

Segmentation of a given image is one of the basic tasks in image analysis. Roughly speaking, an image segmentation is a partition of the image into several parts according to certain characteristics: pixels belonging to the same part all share the same characteristics while neighboring parts demonstrate significant differences in their characteristics. A segmentation of image can be achieved through contours that enclose regions in their interior (plus a background part for pixels not belonging to any particular part). Segmentation techniques can be used to extract objects from an image. There are many algorithms for segmentation. We will concentrate on a popular one that relies on contour propagation.

Representing the contour in an image by $\mathbf{v}(s) = (x(s), y(s))$, we can write its energy functional as

$$E^*_{\text{snake}} = \int_0^1 E_{\text{snake}}(\mathbf{v}(s))ds = \int_0^1 E_{\text{int}}(\mathbf{v}(s)) + E_{\text{image}}(\mathbf{v}(s)) + E_{\text{con}}(\mathbf{v}(s))ds$$

where E_{int} represents the internal energy due to bending, E_{image} gives rise to the image force, and E_{con} gives rise to the external constraint forces.

Before we get into the details of the functional, let's take a closer look at the method. We start with the necessary background material.

3 A Detailed Study of the Method

Now we discuss the details of the method.

- Integration by parts
- Partial derivatives
- Chain rules
- Curves, tangent, normal, and arclength
- Implicit curves (Implicit Function Theorem)
- The shortest path between two points
- Euler-Lagrange equation
- Steepest descent method
- Finite difference method
- Discretization: before or after?

Project IV

This is an individual project. You are welcome to discuss with your fellow students but the final report must be done on your own.

You are also welcome to meet with me to discuss any particular point in the paper.

Theory Part: Please give a brief summary of the method of diffusion maps and the role of the kernel function in the method.

Application Part: You are asked to detect the "true dimension" of the data sets. For convenience, you will be asked to generate the data sets by yourself as instructed (instead of downloading them from the web). The procedure to generate the data sets is a typical one and you may use it in the future when you design any numerical experiments.

Numerical experiment I: To generate the first data set, please do the following.

1. Generated 1000 points: t_i $(i = 1 : 1000)$ that are uniformly distributed random numbers from the interval $[0, 10]$

2. Use the relation:

$$\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = t_i \begin{pmatrix} 3 \\ 2 \\ 1.4 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} + \text{noise}$$

   where the noise is normally distributed with zero mean and unit variance.

3. Plot the points in two figures (as illustrated in Figure 2 below): The right plot is color coded according to the index i which is depicted by the color bar next to the plot. This shows that the points were not ordered as they are generated.

4. Apply the diffusion map algorithm and find the six most important eigenvectors.

5. Apply a PCA algorithm to find the six largest eigenvectors corresponding to the six principal components of the data.

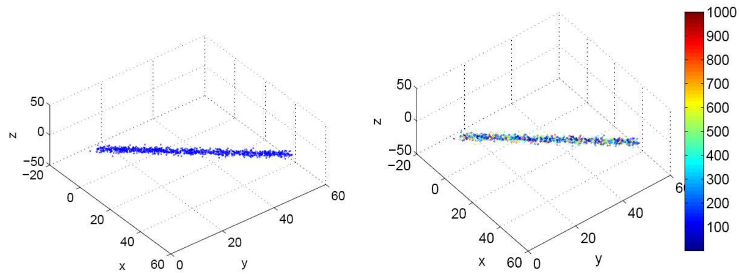

Figure 2: Example plot using color for indexing

6. Plot the points again but this time color them according to the second largest eigenvalue from the diffusion maps and the first eigenvalue from PCA (why?).

7. Discuss your results.

Numerical experiment II: To generate the second data set, please do the following.

1. Generated 1000 points: t_i $(i = 1 : 1000)$ that are uniformly distributed random numbers from the interval $[0, 1]$

2. Use the relation:

$$\begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} = (1 + s_i) \begin{pmatrix} \sin s_i \\ \cos s_i \\ \sin 2s_i \end{pmatrix} + \text{noise}$$

   where $s_i = 10(1 - t_i)^{1.5}$ and the noise is normally distributed with zero mean and unit variance.

3. Plot the points in two figures (as in Experiment I above) using a color bar.

4. Apply the diffusion map algorithm and find the six most important eigenvectors.

5. Apply a PCA algorithm to find the six largest eigenvectors corresponding to the six principal components of the data.

6. Plot the points again but this time color them according to the second largest eigenvalue from diffusion maps and the first eigenvalue from PCA.

7. Discuss the differences in the plots.

8. Use additional plots to make your case.